A study on exotic option in a jump-diffusion with switching type volatility

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Abstract
Fundamental progress has been made in developing more realistic option pricing models. These models has been investigated for standard vanilla options, it is still unknown how well these generalizations improve the exotic options. Using different barrier options on the Data Analysis Expressions, we examine switching type volatility, jump diffusion and a mixed model. This paper, deals with the question of model selection that is nowadays of great importance because of the growing number of models and exotic products.

Keywords: Exotic Options, Switching Type Volatility, Jump Diffusion, Martingale Measure, Equities

1 Introduction
Besides the standard vanilla options, exotic options such as barrier options, look back options, floating-strike options and cliquet options have become very popular financial trading instruments [5]. Unlike the vanilla options, the pay-off functions of the exotic options are path-dependent and hence the problems of obtaining closed form solutions for such options are very much complicated [10]. Several studies have been made in the last few decades in obtaining pricing formulas for the exotic options.

However, not much work has been done for pricing exotic in stochastic volatility models. Accordingly, we present the problem of pricing a clique option when the underlying asset price satisfies a jump-diffusion equation and the volatility changes according to the occurrence of the jumps in the asset price [3].

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2. Financial market model

Let T be a positive constant representing the time horizon and let the market consist of a risk-free asset (bond) and a risky asset (stock). At time t, let the price of the bond be $B_t$ and that of the stock be $S_t$. We assume that $B_0 = 1$ and $S_0 = A$. Let $B_t$ satisfy the equation

$$dB_t = rB_t dt,$$  \hfill (1)

Where $r$ is a positive constant representing the risk-free interest rate of the bond price.

Then $B_t = e^{rt}$. We assume that the price $S_t$ of the risky asset satisfies the stochastic jump-diffusion differential equation

$$dS_t = \left\{ \begin{array}{ll}
\mu dt + \sigma_1 dW_t + \gamma dN_t, & 0 \leq t \leq T_c, \\
\mu dt + \sigma_2 dW_t + \gamma dN_t, & t \geq T_c
\end{array} \right. \hfill (2)$$

where we have assumed

(i) $\mu, \sigma_1, \sigma_2$ and $\gamma$ are positive constants;

(ii) $W_t$ is a standard Brownian motion on a probability space $(\Omega^W, F^W, P^W)$;

(iii) $N_t$ is Poisson process on a probability space $(\Omega^N, F^N, P^N)$;

(iv) $T_c$ is random variable independent of $W_t$ and $N_t$;

(v) $W_t$ and $N_t$ are independent of each other.

Let the Poisson process $N_t$ be of constant intensity $\lambda$ and the probability density function of $T_c$ be $f(u)$. Let $\{F^W_t\}_{0 \leq t \leq T}$ be the natural filtration generated by $W_t$ and $\{F^N_t\}_{0 \leq t \leq T}$ be that generated by $N_t$. Let $(\Omega, F, P)$ be the product space formed by $(\Omega^W, F^W, P^W)$ and $(\Omega^N, F^N, P^N)$. Let $\{F^W_t\}_{0 \leq t \leq T}$ be the filtration generated by the direct product of $\{F^W_t\}_{0 \leq t \leq T}$ and $\{F^N_t\}_{0 \leq t \leq T}$. Then the asset process $\{S^W_t\}_{0 \leq t \leq T}$ is defined on $(\Omega, F, P)$. The equation (2) can be written as

$$\frac{dS_t}{S_t} = \mu dt + \sigma_1 dW_t + \gamma dN_t, 0 \leq t \leq T_c \hfill (3)$$

where $\sigma_t = \sigma_1 1_{\{0 \leq t < T_c\}} + \sigma_2 1_{\{t \geq T_c\}}$ with $1_A$ denoting the indicator function of the set $A$ defined by
1. \( I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \)

3. An equivalent martingale measure and the stock price.

We now proceed to solve the equation (3) for the asset price \( S_t \) and obtain an equivalent martingale measure \( Q \) under which the discounted asset price becomes a martingale. For this, we let \( M_t = N_t - \lambda t \). Then the equation (3) becomes

\[
\frac{dS_t}{S_t} = (\mu + \gamma \lambda) dt + \sigma_t dW_t + \gamma dM_t, 0 \leq t \leq T. (4)
\]

Define the discounted price of the asset

\[
S_t = \frac{S_t}{B_t} = e^{-rt} S_t.
\]

Then equation (4) becomes

\[
\frac{dS_t}{S_t} = (\mu - r + \gamma \lambda) dt + \sigma_t dW_t + \gamma dM_t, 0 \leq t \leq T. (5)
\]

Although \( W_t \) and \( M_t \) are \( P \)-martingales in the above equation (5), the process \( S_t \) is not a \( P \)-martingale.

We now seek a probability measure \( Q \) such that \( S_t \) is a \( Q \)-martingale. First, for each pair of constants \( \theta \) and \( \psi \) with \( \psi \geq 0 \), we define a process \( L_t \) defined by

\[
L_t = \exp \left[ \left( \lambda (1 - \psi) - \frac{1}{2} \theta^2 \right) t + \theta W_t + (\log \psi) N_t \right]. (6)
\]

Then, we have

\[
E^{(P)}[L_t] = e^{\lambda(1-\psi)\theta t - \frac{1}{2} \theta^2 t} e^{\frac{1}{2} \theta^2 t} E \left[ e^{(\log \psi) N_t} \right]
= e^{\lambda(1-\psi)\theta t - \frac{1}{2} \theta^2 t} \sum_{n=0}^{\infty} e^{\log \psi} \frac{(-\lambda \psi)^n}{n!}
= e^{\lambda(1-\psi)\theta t} \sum_{n=0}^{\infty} e^{-\lambda \psi \frac{n!}{n!}}
= e^{\lambda(1-\psi)\theta t} e^{-\lambda(1-\psi)\psi} = 1.
\]
For each $A \in \mathcal{F}_t$, we define a set function $Q$ such that

$$Q(A) = \int_A L_t dP(\omega).$$

Then $Q$ is a probability measure equivalent to $P$ such that the Random-Nikodym derivative

$$\frac{dQ}{dP} = L_t.$$ 

We define

$$W^{(Q)}_t = W_t - \theta t, \quad M^{(Q)}_t = N_t - \lambda \psi t.$$ 

Then, with respect to the measure $Q$, we note that

$$E^{(Q)}[W^{(Q)}_t] = E^{(Q)}[W_t - \theta t] = E^{(Q)}[W_t - \theta t] L_t \Delta t$$

$$= E^{(Q)}[(W_t - \theta t) \exp \left\{ \lambda (1 - \psi) - \frac{1}{2} \theta^2 \right\} t + \theta W_t + (\log \psi) N_t]$$

$$= E^{(Q)}[(W_t - \theta t) e^{\frac{1}{2} \theta^2} e^{\theta W_t}]$$

$$= E^{(Q)}[\frac{\partial}{\partial \theta} e^{\frac{1}{2} \theta^2}]$$

$$E^{(Q)}[W^{(Q)}_t] = \frac{\partial}{\partial \theta} E^{(P)}\left[ e^{\frac{1}{2} \theta^2} \right] = \frac{\partial}{\partial \theta} (1) = 0 \quad (7)$$

Similarly, we note that
\[ E^{(Q)} \left[ M^{(Q)}_t \right] = E^{(Q)} \left[ N_t - \lambda \psi t \right] = E^{(Q)} \left[ \{ N_t - \lambda \psi t \} L_t \right] \\
= E^{(Q)} \left[ \{ N_t - \lambda \psi t \} \exp \left\{ \left( 1 - \psi \right) - \frac{1}{2} \theta^2 \right\} t + \theta W_t + (\log \psi) N_t \right] \\
= E^{(Q)} \left[ \{ N_t - \lambda \psi t \} e^{\lambda \left( 1 - \psi \right) t} \psi^{N_t} \right] \\
= e^{\lambda \left( 1 - \psi \right) t} E^{(P)} \left[ N_t \psi^{N_t} \right] - \lambda \psi t E^{(P)} \left[ e^{\lambda \left( 1 - \psi \right) t} \psi^{N_t} \right] \\
= e^{\lambda \left( 1 - \psi \right) t} \sum_{n=0}^{\infty} \psi^n \sum_{k=0}^{\infty} \frac{(\lambda \psi t)^n}{n!} - \lambda \psi t e^{\lambda \left( 1 - \psi \right) t} \sum_{n=1}^{\infty} \frac{(\lambda \psi t)^n}{n!} \\
= e^{\lambda \left( 1 - \psi \right) t} \lambda \psi t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda \psi t)^{n-1}}{(n-1)!} - \lambda \psi t e^{\lambda \left( 1 - \psi \right) t} \sum_{n=1}^{\infty} \frac{(\lambda \psi t)^n}{n!} \\
= E^{(Q)} \left[ M^{(Q)}_t \right] = \lambda \psi t - \lambda \psi t = 0 (8) \]

Further, the process \( W^{(Q)}_t \) is a standard Brownian motion with respect to the measure \( Q \).

The equation (5) becomes

\[
\frac{d S_t}{S_t} = \left( \mu - r + \gamma \lambda \right) dt + \sigma d \left( W^{(Q)}_t + \theta t \right) + \gamma d \left( M^{(Q)}_t + \lambda \psi t - \lambda t \right) \\
= \left\{ \mu - r + \sigma, \theta + \gamma \lambda \psi \right\} dt + \sigma d W^{(Q)}_t + \gamma d M^{(Q)}_t \\
(9)
\]

We eliminate the presence \( \mu \) in (9) by choosing \( \theta \) and \( \psi \) such that

\[
\mu - r + \sigma, \theta + \gamma \lambda \psi = 0 \quad (10)
\]

Then, from the equation (10). We obtain

\[
\theta = \frac{r - (\mu + \gamma \lambda \psi)}{\sigma}. \quad (11)
\]

Then the equation (9) yields

\[
\frac{d S_t}{S_t} = \sigma d W^{(Q)}_t + \gamma d M^{(Q)}_t. \quad (12)
\]
The equation (12) clearly establishes the fact that $\tilde{S}_t$ is a $Q$–martingale. We solve the equation (12) by putting $X_t = \log S_t$ and noting the fact that

$$dW^{(Q)}_t = 0 \left( \sqrt{dt} \right), \left\{ dW^{(Q)}_t \right\}^2 = dt, dM^{(Q)}_t = dN_t - \lambda \psi dt, \left\{ dM^{(Q)}_t \right\}^n = dN_n, n = 2, 3,...$$

We obtain from the equation (12) that

$$dX_t = \log \left\{ 1 + \frac{d\tilde{S}_t}{\tilde{S}_t} \right\}$$

$$= d\tilde{S}_t \cdot \frac{1}{\tilde{S}_t} \left\{ d\tilde{S}_t \right\}^2 + \frac{1}{3} \left\{ d\tilde{S}_t \right\}^3 - ...$$

$$= \sigma_t dW^{(Q)}_t + \gamma dM^{(Q)}_t - \frac{1}{2} \sigma_t^2 dt + \frac{1}{3} \gamma^3 dN_t - \frac{1}{4} \gamma^4 dN_t + ...$$

$$dX_t = \left\{ \frac{1}{2} \sigma_t^2 + \lambda \gamma \right\} dt + \sigma_t dW^{(Q)}_t + \left\{ \log \left( 1 + \gamma \right) \right\} dN_t, (13)$$

When $0 \leq t \leq T_c$, we have from the equation (13)

$$X_t = X_0 - \left\{ \frac{1}{2} \sigma_t^2 + \lambda \gamma \right\} t + \sigma_t W^{(Q)}_t + \left\{ \log \left( 1 + \gamma \right) \right\} N_t$$

That is, we have

$$\log \tilde{S}_t = \log S_0 - \left\{ \frac{1}{2} \sigma_t^2 + \lambda \gamma \right\} t + \sigma_t W^{(Q)}_t + \left\{ \log \left( 1 + \gamma \right) \right\} N_t, (14)$$

Similarly, when $T_c \leq t \leq T$, we have from the equation (13)

$$\log \tilde{S}_t = \log S_{t_c} - \left\{ \frac{1}{2} \sigma_t^2 + \lambda \gamma \right\} \left( t - T_c \right) + \sigma_t \left\{ W^{(Q)}_t - W^{(Q)}_{t_c} \right\} + \left\{ \log \left( 1 + \gamma \right) \left\{ N_t - N_{t_c} \right\} \right\}, (15)$$

Using (14) and (15), we proceed to solve the problem of pricing a clique option

4. Conclusion.

This paper deals with the question of model selection and exotic products. As the paper demonstrates, the evolution of the different barrier options on the Data Analysis Expressions, we examine switching type volatility, jump diffusion and a mixed model. The two models differ significantly in their pricing of a range of standard exotic option contracts. While this paper has mainly focused on equities, we point out that developed methodology works equally well for Data Analysis Expressions rate.

5. References


