Abstract

We introduce a new concept known as Hop graph of a graph. Hop graph suggests an easy way to determine the hop domination number of a given graph. We have determined Hop graphs of paths $P_n$, cycles $C_n$ and wheels $W_n$. We have also proved that hop graph $H(G)$ for any graph $G$ is not equal to Path $P_4$. This paper presents some elementary but important properties of hop graph.

1. INTRODUCTION

S.K. Ayyaswamy and C. Natarajan [1] have determined hop domination number for paths $P_n$, cycle $C_n$, complete graphs, complete bipartite graphs and wheel graphs. Concept of hop graph of a graph is introduced. Many illustrations are given to explain how to use hop graphs for finding hop domination number. It is seen that if forms a middle step in determination of hop domination numbers.

2. DEFINITION

The hop graph $H(G)$ of a graph $G$ is the graph obtained from $G$ by taking $V(H(G))=V(G)$ and joining two vertices $u,v$ in $H(G)$ if and only if they are at a distance 2 in $G$.

2.1 Example

![Hop Graph Example](image-url)
2.3 Theorem  Let G be a graph with diameter 2. Then H(G) = $\overline{G}$

Proof:- The hop graph H(G) of a graph G is the graph obtained from G with

V(H(G))=V(G) and
joining two vertices u,v in H(G) if and only if they are at a distance 2 in G.

\[ V(\overline{G})=V(G)=V(H(G)) \]

u,v are adjacent in H(G) $\iff$ $d_G(u,v)=2$.

$\Rightarrow$ u,v are not adjacent in G.
$\Rightarrow$ u,v are adjacent in $\overline{G}$.
$\Rightarrow$ $d_{\overline{G}}(u,v)=1$.

Hence $H(G) \supseteq \overline{G}$. .....(1)

Conversely, u,v are not adjacent in G $\iff$ $d_{\overline{G}}(u,v)=1$.

$\Rightarrow$ u,v are adjacent in $\overline{G}$.
$\Rightarrow$ u,v are not adjacent in G.
$\Rightarrow$ $d_G(u,v)=2$.

Hence $\overline{G} \supseteq H(G)$ .....(2)

From (1) and (2), Hence $H(G) = \overline{G}$.

2.4 Note
1. Full degree vertex will become isolated in its hop graph.
2. Hop graph of a complete graph is a null graph.

3. DETERMINATION OF HOP DOMINATION NUMBER USING HOP GRAPH

3.1 Theorem  $\gamma_h(G)= \gamma(H(G))$

Proof:- Let D be a hop dominating set of G.

For any vertex u in V\D, there exists a vertex v in D such that $d_G(u,v)=2$.

Therefore, for any vertex u in V\D, there exists v in D such that $d_{H(G)}(u,v)=1$.

Hence D is a dominating set of H(G).

Therefore, $\gamma_h(G) \geq \gamma(H(G))$ ......(1)

Conversely, Let D be a dominating set of H(G).

For any vertex u in V\D, there exists a vertex v in D such that $d_{H(G)}(u,v)=1$.

Therefore, for any vertex u in V\D, there exists v in D such that $d_G(u,v)=2$.

Hence D is a hop dominating set of G.

Therefore, $\gamma_h(G) \leq \gamma(H(G))$ ......(2)

From (1) and (2), $\gamma_h(G)= \gamma(H(G))$

3.2 Note: The above theorem gives an easy way to determine hop domination number of any graph. First construct the hop graph H(G) of the given graph G and then determine the domination number of H(G).

3.3 Illustration  Graph of Octahedron
H(G) is the disjoint union of three copies of $P_2$.

$\gamma (H(G)) \geq 3$.

$S=\{1,2,3\}$ is a minimum dominating set of $H(G)$.

Hence $S$ is a minimum hop dominating set of $G$ and $\gamma_h(G)=3$.

**3.4 Illustration Graph of Petersen**

Minimum hop dominating set of $G$, $S=\{2,7\}$

Minimum dominating set of $H(G)$, $S=\{2,7\}$

**3.5 Illustration Graph of Tetrahedron**

H(G) is the disjoint union of four copies of $K_1$.

$S=\{1,2,3,4\}$ is a minimum dominating set of $H(G)$.

Hence $S$ is a minimum hop dominating set of $G$ and $\gamma_h(G)=4$.

**3.6 Illustration Graph of Dodehedron**
Hop dominating set of G, S={1,6,11,19}
Dominating set of H(G), S={1,6,11,19}

3.7 Illustration  Graph of Icosahedron

Hop dominating set of G, S={1,2,7}
Dominating set of H(G), S={1,2,7}

3.8 Theorem

1. $H(P_n) = \begin{cases} P_{n/2} \cup P_{n/2} & \text{if } n \text{ is even} \\ P_{n/2} \cup P_{n/2} & \text{if } n \text{ is odd} \end{cases}$

2. $\gamma_h(P_n) = \begin{cases} 2r & \text{if } n = 6r \\ 2r + 1 & \text{if } n = 6r + 1 \\ 2r + 2 & \text{if } n = 6r + s; 2 \leq s \leq 5 \end{cases}$

Proof:

i) $H(P_n)$ is the disjoint union of 2 paths, one of the length $\left[\frac{n}{2}\right]$ and other of length $\left[\frac{n}{2}\right]$. Therefore, $H(P_n) = \begin{cases} P_{n/2} \cup P_{n/2} & \text{if } n \text{ is even} \\ P_{n/2} \cup P_{n/2} & \text{if } n \text{ is odd} \end{cases}$

Case i) $n=6r$

$\left[\frac{n}{2}\right] = 3r = \left[\frac{n}{2}\right]$

$\gamma (H(P_n)) = \gamma (P_{3r}) + \gamma (P_{3r})$
\[ \gamma(H(P_n)) = \gamma(P_{3r+2}) + \gamma(P_{3r+2}) = 2 \frac{3r + 2}{3} = 2r + 2. \]

Case ii) \( n = 6r + 1 \)

\[ \left\lfloor \frac{n}{2} \right\rfloor = 3r + 1 \quad \text{and} \quad \left\lceil \frac{n}{2} \right\rceil = 3r. \]

\[ \gamma(H(P_n)) = \gamma(P_{3r+1}) + \gamma(P_{3r}) \]

\[ = \left\lfloor \frac{3r + 1}{3} \right\rfloor + \left\lfloor \frac{3r}{3} \right\rfloor = (r+1) + r = 2r + 1. \]

Case iii) \( n = 6r + 2 \)

\[ \left\lfloor \frac{n}{2} \right\rfloor = 3r + 1 = \left\lceil \frac{n}{2} \right\rceil \]

\[ \gamma(H(P_n)) = \gamma(P_{3r+1}) + \gamma(P_{3r+1}) \]

\[ = (r+1) + (r+1) = 2r + 2. \]

Case iv) \( n = 6r + 3 \)

\[ \left\lfloor \frac{n}{2} \right\rfloor = 3r + 2 \quad \text{and} \quad \left\lceil \frac{n}{2} \right\rceil = r + 1. \]

\[ \gamma(H(P_n)) = \left\lfloor \frac{3r + 2}{3} \right\rfloor + \left\lfloor \frac{3r + 1}{3} \right\rfloor = (r+1) + (r+1) = 2r + 2. \]

Case v) \( n = 6r + 4 \)

\[ \left\lfloor \frac{n}{2} \right\rfloor = 3r + 2 = \left\lceil \frac{n}{2} \right\rceil \]

\[ \gamma(H(P_n)) = \gamma(P_{3r+2}) + \gamma(P_{3r+2}) \]

\[ = 2 \left\lfloor \frac{3r + 2}{3} \right\rfloor = 2r + 2. \]

Case vi) \( n = 6r + 5 \)
\[ \left\lfloor \frac{n}{2} \right\rfloor = 3r+3 \text{ and } \left\lceil \frac{n}{2} \right\rceil = 3r+2. \]

\[ \gamma(H(P_n)) = \gamma(P_{3r+3}) + \gamma(P_{3r+1}) \]

\[ = \left\lfloor \frac{3r+3}{3} \right\rfloor + \left\lceil \frac{3r+3}{3} \right\rceil = (r+1) + (r+1) = 2r+2. \]

Hence \( \gamma_h(P_n) = \gamma(H(P_n)) = \begin{cases} 2r & \text{if } n = 6r \\ 2r+1 & \text{if } n = 6r+1 \\ 2r+2 & \text{if } n = 6r+s; 2 \leq s \leq 5 \end{cases} \)

**3.9 Theorem**

\[ H(C_n) = \begin{cases} C_n \cup C_n & \text{if } n \text{ is even} \\ C_n & \text{if } n \text{ is odd} \end{cases} \]

\[ \gamma_h(C_n) = \begin{cases} 2r & \text{if } n = 6r \\ 2r+1 & \text{if } n = 6r+1 \\ 2r+2 & \text{if } n = 6r+s; 2 \leq s \leq 5 \end{cases} \]

**Proof:-**

1. \( H(C_n) \) is the disjoint union of 2 cycles.

Therefore, \( H(C_n) = \begin{cases} C_n \cup C_n & \text{if } n \text{ is even and } n \text{ is odd} \\ C_n & \text{if } n \text{ is odd} \end{cases} \)

2. **Case i**) \( n=6r \)

\[ \gamma(H(C_n)) = 2 \gamma(C_{3r}) = 2 \left\lfloor \frac{3r}{3} \right\rfloor = 2r. \]

**Case ii**) \( n=6r+1 \)

\[ \gamma(H(C_n)) = \gamma(C_{3r}) = \left\lceil \frac{6r+1}{3} \right\rceil = 2r+1. \]

**Case iii**) \( n=6r+2 \)

\[ \gamma(H(C_n)) = 2 \gamma(C_{3r+1}) = 2 \left\lceil \frac{6r+1}{3} \right\rceil = 2(r+1) = 2r+2. \]

**Case iv**) \( n=6r+3 \)
$$\gamma (H(C_n)) = \gamma (C_{6r+3}) = \left\lceil \frac{6r+3}{3} \right\rceil = 2r+1.$$ 

**Case v)** \(n=6r+4\) 
$$\gamma (H(C_n)) = 2\gamma (C_{3r+2}) = \left\lceil \frac{3r+2}{3} \right\rceil = 2(r+1) = 2r+2.$$ 

**Case vi)** \(n=6r+5\) 
$$\gamma (H(C_n)) = \gamma (C_{6r+5}) = \left\lceil \frac{6r+5}{3} \right\rceil = 2r+2.$$ 

Hence, 
$$\gamma_h (C_n) = \gamma (H(C_n)) = \begin{cases} 
2r & \text{if } n = 6r \\
2r+1 & \text{if } n = 6r+1 \text{ or } 6r+3 \\
2r+2 & \text{if } n = 6r + s; s = 2, 4, 5
\end{cases}$$

### 3.10 Note
In paper [1] it has been stated as 
$$\gamma_h (C_n) = \begin{cases} 
2r & \text{if } n = 6r \\
2r+1 & \text{if } n = 6r+1 \\
2r+2 & \text{if } n = 6r + s; 2 \leq s \leq 5
\end{cases}$$

But our result, 
$$\gamma_h (C_n) = \begin{cases} 
2r & \text{if } n = 6r \\
2r+1 & \text{if } n = 6r+1 \text{ or } 6r+3 \\
2r+2 & \text{if } n = 6r + s; s = 2, 4, 5
\end{cases}$$

It differs from the previous result in the case \(n=6r+3\). We can verify that our result is true by the following illustration \(W_9\).

### 3.11 Illustration
Consider \(C_9\), 

\[n = 9 = (6 \times 1) + 3\]

\[S = \{3, 6, 9\}\] is a hop dominating set of \(G\). 

$$\gamma_h (C_9) = 2r+1.$$ 

In general, for \(C_{6r+3}\), minimum hop dominating set is the same as the minimum dominating set. 

$$\gamma (C_{6r+3}) = \left\lceil \frac{6r+3}{3} \right\rceil = 2r+1.$$ 

### 3.12 Definition
Wheel graph \((W_n)\) of size \(n\) is the graph with \(V(W_n) = \{v_0, v_1, v_2, \ldots, v_n\}\) and
E(W_n) = \{v_0, v_i/=1,2,...,n\} \cup \{v_i,v_{i+1}/i=1,2,...,n-1\} \cup \{v_nv_1\}.

3.13 Theorem
1. \(H(W_n) = K_1 \cup G_n\) where \(G_n\) is a n-2 regular graph on n vertices.
2. \(\gamma(W_n) = 3\) for \(n \geq 4\).

Proof:-
1) \(H(W_n)\) is the disjoint union of \(K_1\) and \(G_n\) where \(G_n\) is a n-2 regular graph on n vertices.

\[W_5, H(W_5), W_6, H(W_6)\]

2) \(|V(W_n)| = n\) and \(\Delta(W_n) = n-2\).

Therefore, \(\gamma(W_n) \geq \frac{n}{n-2} + 1 = \frac{n}{n-1}\).

\(\gamma(H(W_n)) = 1 + \gamma(W_n)\)

\(\geq 1 + \frac{n}{n-1}\).

\(\geq 3\) (since \(\frac{n}{n-1} > 1\)).

In \(H(W_n)\), \(v_0\) is an isolated vertex and \(v_i\) is adjacent to all \(v_i\)'s except \(v_{i+1}\) and \(v_{i-1}\).

In particular, \(v_1\) is adjacent to \(v_3,v_4,...,v_{n-1}\) and \(v_2\) is adjacent to \(v_4,v_5,...,v_n\) in \(H(W_n)\).

Therefore, \(\{v_0,v_1,v_2\}\) is a hop dominating set of \(H(W_n)\).

Hence, it is a dominating set of \(W_n\).

Therefore, \(\gamma(H(W_n)) = 3\) for \(n \geq 4\).

3.14 Definition [2]
The windmill graph \(W_d(k,n)\) is an undirected graph constructed for \(k \geq 2\) and \(n \geq 2\) by joining \(n\) copies of the complete graph \(K_k\) at a shared vertex.
3.15 Theorem \( \gamma (H(W_d(k,n)))=3 \) for \( k \geq 2, n \geq 2 \).

**Proof:** Let \( u_0 \) denote the shared vertex and \( \{v_{ij} | i \leq j \leq n-1\} \) denote the vertices in the \( i^{th} \) copy of \( K_k \) for \( 1 \leq i \leq n \).

The hop graph \( H(G) \) will be the disjoint union of \( K_1 \) and a complete n-partite graph \( K_{n-1,n-2,\ldots,n-1}=G' \) (say).

The \( n \) partition of \( G' \) will be \( \{a_{i1},a_{i2},a_{i3},\ldots\ldots,a_{ik-1}\} \) for \( 1 \leq i \leq n \).

\( u_0 \) is isolated vertex, so \( u_0 \) should be contained in any dominating set.

We know that, \( \gamma (G')=2. \) (since \( G' \) is complete n-partite graph)

Hence \( \gamma (H(W_d(k,n)))=2+1=3 \) and so \( \gamma (H(W_d(k,n)))=3 \).

Therefore, \( \gamma (H(W_d(k,n)))=3 \).

3.16 Theorem If \( G' \) is a graph with a full degree vertex, then \( G' \) cannot be the hop graph of any graph.

**Proof:** Let \( G' \) be a \((p,q)\) graph with a full degree vertex \( v \).

If possible, let \( G' \) be the hop graph of a graph \( G \).

\( \Rightarrow \) \( \gamma (G'=(H(G)) \)

\( d_G(v)=p-1 \Rightarrow d_G(u,v)=2 \) for all \( u (\neq v) \) in \( V(G)=V(G') \)

\( \Rightarrow \) (since \( d_G(u_0,v)=2 \) \Rightarrow there is a \( u_0-v \) path of length 2.

\( \Rightarrow \) there is a vertex \( v' \) such that \( u_0 \) \( v'v \) is a path in \( G \).

\( \Rightarrow d_G(v',v)=1. \)

Hence \( G' \) cannot be the hop graph of any graph \( G \).

3.17 Theorem \( \omega (H(G)) \geq \omega (G) \) where \( \omega \) denotes the number of components.

**Proof:** Two vertices are connected in \( H(G) \) only when they are connected in \( G \).

Therefore, Two vertices belong to the same component of \( H(G) \) only when they belong to the same component of \( G \).

Hence \( \omega (H(G)) \geq \omega (G) \).

3.18 Theorem \( P_4 \neq H(G) \) for any graph \( G \).

**Proof:** If possible let \( P_4=H(G) \) for some graph \( G \).

\( P_4 \) is connected \( \Rightarrow \) \( G \) is connected.

\( G \) should have four vertices.

Let \( P_4 \) be labelled as follows.

\( V_1 \) is not adjacent \( v_1 \) and \( v_3 \) in \( G \).

Therefore, \( d_G(v_2)=1. \)

\( V_2 \) is not adjacent \( v_2 \) and \( v_4 \) in \( G \).

Therefore, \( d_G(v_3)=1. \)

\( V_3 \) is not adjacent \( v_3 \) in \( G \).

Therefore, \( d_G(v_1) \leq 2. \)

\( V_4 \) is not adjacent \( v_4 \) in \( G \).

Therefore, \( d_G(v_4) \leq 2. \)
These four conditions are not satisfied by hop graph of any graph G on four vertices. Therefore, $P_4 \neq H(G)$.

4. CONCLUSION

The short analysis carried out in this paper made us understand that there are so many openings in this area of research. The concept of hop graph is simple but it is a source of inspiration. In this paper it has been used merely as a middle step. It opens many avenues for researchers. Graphs which can be realized as hop graphs can be characterized. Properties of the given graph carried over to its hop graph may be studied.

5. REFERENCES


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