Abstract
Prime labeling was introduced by T.Deretsky, S.M.Lee, and J.Mitchem in 1991\[1\] and [2]. Total Prime labeling was introduced by M.Ravi @ Ramasubramanian and R.Kala in 2012\[3\]. A graph \( G(V,E) \) is said to have a total prime labeling if its vertices and edges can be labeled with distinct integers from \( \{1,2,..., p+q\} \) such that for each vertex of degree at least 2, the greatest common divisor of the labels on its incident edges is 1. This paper introduces a slightly modified version of the above said labeling. A highly total prime labeling of a graph \( G \) is an injective function \( f: V \cup E \rightarrow \) \( \{1,2,3,..., p+q\} \) such that any pair of adjacent edges receive relatively prime labelings. Properties of this labeling are studied and some common families such as paths, cycles, coronas, two cycles with one common edge are proved to be highly total prime.

1. INTRODUCTION
Total Prime labeling was introduced by M.Ravi @ Ramasubramanian and R.Kala in 2012\[3\]. We introduce a slightly modified version of the above said labeling as highly total prime labeling. Some Common families such as paths, cycles, coronas, and two cycles with one common edge are proved to be highly total prime. Some other families of graphs have been proved not to be highly total prime.

2. DEFINITION
Let \( G=(V, E) \) be a graph with \( p \) vertices and \( q \) edges. A bijection \( f: V \cup E \rightarrow \{1,2,3,...,p+q\} \) is said to be a Highly total prime labeling if

(i) for each edge \( e = uv \), the labels assigned to \( u \) and \( v \) are relatively prime.
(ii) Any pair of adjacent edges receives relatively prime labeling. A graph which admits highly total prime labeling is called Highly Total Prime Graph.

2.1 Example

$C_4$ is a Highly Total Prime Graph

2.2 Definition

A set of points in $G$ is independent if no two of them are adjacent. The largest number of points in such a set is called the point independence number of $G$ and is denoted by $\beta_0(G)$ or $\beta_0$.

2.3 Example

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$S = \{v_1, v_3, v_5\}$ is a maximum independent set.

Independence number of $G$ is 3.

2.4 Definition

An independent set of lines of $G$ has no two of its lines adjacent and the maximum cardinality of such a set is the line independence number $\beta_1(G)$ or $\beta_1$.

2.5 Example

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

$S = \{e_4, e_5\}$ is a maximum independence set.
Independence number of $G$ is 2.

2.6 Theorem

If a connected graph $G$ is highly total prime, then $\beta_0 + \beta_1 \geq \left\lceil \frac{p+q}{2} \right\rceil$ where $\beta_0$ is the point independence number of $G$, $\beta_1$ is the line independence number of $G$, $p$ is the number of vertices of $G$ and $q$ is the number of edges of $G$.

**Proof:**

Let $G$ be a connected graph which is highly totally prime. Then there is an injective function $f : V \cup E \rightarrow \{1, 2, 3, ..., p+q\}$ such that any two adjacent vertices receive relatively prime labels and any two adjacent edges receive relatively prime labels.

There are $\left\lceil \frac{p+q}{2} \right\rceil$ even integers in $\{1, 2, 3, ..., p+q\}$. The edges with even labels should be a matching of $G$ and the collection of evenly labelled vertices should form an independent set of $G$.

$\beta_0$ is the size of maximum independent set and $\beta_1$ is the size of maximum matching.

Hence $\left\lceil \frac{p+q}{2} \right\rceil \leq \beta_0 + \beta_1$.

2.7 Theorem

Complete graphs $K_n$ ($n \geq 3$) are not highly total prime.

**Proof:**

Let $K_n$ denote complete graph with $n$ vertices.

$p = n, q = \binom{n}{2}$, $\beta_0(K_n) = 1$ and $\beta_1(K_n) = \left\lceil \frac{n}{2} \right\rceil$.

$\beta_0 + \beta_1 = 1 + \left\lceil \frac{n}{2} \right\rceil \leq 1 + \frac{n}{2} \rightarrow (1)$

$p+q = n + \frac{n(n-1)}{2} = \frac{2n+n^2-n}{2} = \frac{n(n+1)}{2}$

$\left\lceil \frac{p+q}{2} \right\rceil = \left\lceil \frac{n(n+1)}{4} \right\rceil \rightarrow (2)$

$\left\lceil \frac{p+q}{2} \right\rceil -(\beta_0 + \beta_1) \geq \left\lceil \frac{n(n+1)}{4} \right\rceil -(1+\frac{n}{2}) > 0$ for $n \geq 3$.

Therefore, for $n \geq 3$, $K_n$ is not highly total prime.

2.8 Note

$K_2$ is highly total prime.
2.9 Theorem
K_{n,n} the complete bipartite graphs are not highly total prime for n > 2.

Proof:
Let K_{n,n} denote the complete bipartite graphs of size n.
Then p = n+n = 2n, q = n×n = n^2
\beta_0(K_{n,n}) = n and \beta_1(K_{n,n}) = n
\beta_0 + \beta_1 = 2n < \left\lfloor \frac{2n + n^2}{2} \right\rfloor = \left\lfloor \frac{p+q}{2} \right\rfloor \text{ for } n \geq 3.
Hence the result.

2.10 Note
K_{2,2} is highly total prime.

2.11 Theorem:
Odd cycles are not highly total prime.

Proof:
Consider the cycle C_n with n = 2r+1
p(C_n) = 2r+1, q(C_n) = 2r+1
\beta_0(C_n) = \left\lfloor \frac{2r+1}{2} \right\rfloor = r and \beta_1(C_n) = \left\lfloor \frac{2r+1}{2} \right\rfloor = r
\beta_0 + \beta_1 = 2r, \frac{p+q}{2} = 2r+1
Hence \beta_0 + \beta_1 < \frac{p+q}{2}
So C_n is not highly total prime.

3. SOME FAMILIES OF HIGHLY TOTAL PRIME GRAPHS

3.1 Theorem
Paths are highly total prime.

Proof:
Let P_n denote path of length n.
Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$ and
\[E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-2}v_{n-1}, v_{n-1}v_n\}\]
Define $f(v_j) = j$, for $1 \leq j \leq n$
\[f(v_{i+1}) = n+i, \text{ for } i=1,2,\ldots,n\]
Claim: $f$ is a highly total prime labeling.
Adjacent vertices as well as adjacent edges receive consecutive numbers, which are relatively prime.
Therefore, $f$ is a highly total prime labeling.

3.2 Illustration

\[\begin{array}{c}
\begin{array}{cccccccccccc}
\text{V}_1 & 7 & \text{V}_2 & 8 & \text{V}_3 & 9 & \text{V}_4 & 10 & \text{V}_5 & 11 & \text{V}_6
\end{array}\\
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\end{array}\]

3.3 Theorem

Even cycles admit highly total prime labeling.

Proof:
Let $C_{2n}$ denote cycle of length $2n$.
Let $V(C_{2n}) = \{v_1, v_2, v_3, ..., v_{2n}\}$
\[E(C_{2n}) = \{v_1v_2, v_2v_3, ..., v_{2n-1}v_{2n}, v_{2n}v_1\}\]
Define $f(v_j) = j$, for $j = 1,2,\ldots,n$
\[f(v_{i+1}) = i+4, \text{ for } i=1,2,\ldots,n-1\]
\[f(v_1v_n) = n+1\]
Adjacent edges receive consecutive numbers for $i=1, 2, 3, ...,n-1$ and $(1,n) = 1$
Therefore, $f$ is a highly total prime labeling.

3.4 Illustration
\[\begin{array}{c}
\begin{array}{cccc}
\text{V}_1 & 6 & \text{V}_2 & 5
\end{array}\\
\begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}
\end{array}\]

3.5 Theorem

Disjoint union of two even cycles $C_{2n} \cup C_{2n}$ of same length are highly total prime.

Proof:
Case (i): $n = 2$
\[\begin{array}{c}
\begin{array}{cccc}
\text{V}_1 & 15 & \text{V}_z & 10
\end{array}\\
\begin{array}{cccc}
4 & 1 & 9 & 16
\end{array}
\end{array}\]
\[\begin{array}{c}
\begin{array}{cccc}
\text{V}_1 & 11 & \text{V}_z & 6
\end{array}\\
\begin{array}{cccc}
5 & 12 & 8 & 14
\end{array}
\end{array}\]
Case(ii): \( n \geq 3 \)

Consider \( C_{2n} \cup C_{2n} \), the disjoint union of two even cycles of length \( 2n \).

Let \( V(C_{2n} \cup C_{2n}) = \{ u_1, u_2, \ldots, u_{2n-1}, u_{2n}, v_1, v_2, \ldots, v_{2n} \} \)

\[ E(C_{2n} \cup C_{2n}) = \{ u_1 u_2, u_2 u_3, \ldots, u_{2n-1} u_{2n}, u_{2n} u_1, v_1 v_2, v_2 v_3, \ldots, v_{2n-1} v_{2n}, v_{2n} v_1 \} \]

Define \( f(u_i, u_{i+1}) = i+6 \), for \( i=1,2,\ldots,2n-1 \)

\[ f(u_i, u_{i+1}) = 2n, \]

\[ f(v_i, v_{i+1}) = i+2n, \text{ for } i=1,2,\ldots,2n-1 \]

\[ f(v_1, v_{2n}) = 4n \]

Adjacent edges receive consecutive numbers for \( i = 1,2,\ldots,2n-1 \) and \( (1,2n) = 1 \).

In the second cycle adjacent edges receive consecutive numbers for \( i=1,2,\ldots,2n-1 \) except \( v_{2n} v_1 \).

Claim: \( (2n+1, 4n) = 1 \)

If possible, let \( (2n+1, 4n) = d > 1 \). Then \( d \geq 2 \)

Let \( 2n+1 = k_1d, 4n = k_2d \)

2n+1 is odd \( \Rightarrow k_1 \) is odd and \( d \) is odd. Hence \( d \geq 3 \)

4n is even, \( d \) is odd \( \Rightarrow k_2 \) is even

2n+1 = \( k_1d \Rightarrow 2n = k_1d-1 \Rightarrow 4n = 2k_1d-2 \Rightarrow k_2d = 2k_1d-2 \)

\( \Rightarrow (2k_1-k_2)d = 2 \)

\( d \) is not divisible by 2, since \( d \) is odd.

Therefore, \( 2k_1-k_2 = 1 \) (or) \( 2k_1-k_2 \neq 1 \)

\( K_1 \) is odd; \( k_2 \) is even \( \Rightarrow 2k_1-k_2 \neq 1 \)

Hence the claim.

### 3.6 Illustration

![Illustration](image_url)

### 3.7 Theorem:

The corona graphs \( C_n \circ K_1 \) are highly total prime when \( n \equiv 0 \) (mod3).

**Proof:**

Let \( G \) be the corona graphs \( C_n \circ K_1 \).
V(G) = { u_1, u_2, ..., u_n, v_1, v_2, ..., v_n }
E(G) = { u_1 u_2, u_2 u_3, ..., u_n u_1, u_n u_1, v_1, v_2, ..., v_n v_n }

To define labeling f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p+q\} following cases are to be considered.

Case 1: n \equiv 0 \pmod{3}
f(u_j) = 2n+j, f(v_j) = 4n-j+1 for j=1,2,...,n
f(v_j) = 3m-2j, for j=7,8,...,n
f(u_i v_{i+1}) = 2i+1, for i=1,2,...,n
f(u_i u_n) = 1
f(u_i v_i) = 2i, for i=1,2,...,n

v_1, v_2, ..., v_n are pendant vertices d(u_i) = 3, 1 \leq i \leq n

The three edges incident with u_i receives three consecutive integers.
Adjacent edges receive pairwise relatively prime labels.
we can easily verify that (2n+j, 4n-j+1) = 1 and (3n, 2n+1) = 1.
Hence corona graphs C_n \circ K_1 are highly total prime.

3.8 Illustration

3.8 Theorem:
Two cycles of same size with a common edge are highly total prime.

Proof:
Let G be the graph with
V(G) = { v_1, v_2, ..., v_{2n-2} } and
E(G) = { v_i v_{i+1} / i=1,2,...,2n-1 } \cup \{ v_{2n-2}, v_1, v_n v_1 \}
Define f(v_i) = 2n+j, 1 \leq j \leq 2n-3
f(v_1 v_{2n-2}) = 2n-1
f(v_1 v_n) = 1

Edges incident at v_n are labeled with 1, n, n+1.
Edges incident at v are labeled with 1,2,...,2n-1.
d(v_i) = 2 for i \neq 1, n.
Two edges incident at any such vertex are labeled with consecutive integers.
3.9 Illustration

3.10 Definition [4]

Duplication of a vertex v of graph G produces a new graph G’ by adding a new vertex v’ such that N(v’)=N(v). In other words a vertex v’ is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v’ in G.

3.11 Theorem

The graph D(P_n) obtained by duplicating every vertex by a vertex in the path P_n is highly total prime only when n ≤ 4 where n is the number of vertices in P_n.

Proof:

Path P_n has n vertices and n-1 edges
Let D(P_n) denote the graph obtained by duplicating every vertex in P_n by a vertex. D(P_n) has 2n vertices and (n-1)+2+(n-2)2 = n-1+2+2n-4 = 3n-3 edges
β_0 is the size of maximum independent set and β_1 is the size of maximum matching.

In P_n, β_0 = n

Claim: β_1 = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}

Proof for the Claim: when n=2k, β_1 = 2 \leq 2 \left\lfloor \frac{2k-1}{2} \right\rfloor = 2 \left\lfloor k - \frac{1}{2} \right\rfloor = 2k = n.

when n=2k+1, β_1 = 2 \left\lfloor \frac{n-1}{2} \right\rfloor = 2 \left\lfloor \frac{2k + 1 - 1}{2} \right\rfloor = 2 \left\lfloor \frac{2k}{2} \right\rfloor = 2k = n-1

By theorem” If a connected graph G is highly total prime, then β_0 + β_1 ≥ \left\lceil \frac{p+q}{2} \right\rceil” where

β_0 is the point independence number of G, β_1 is the line independence number of G, p is the number of vertices of G and q is the number of edges of G.”

Let n ≥ 5
For D(P_n),

p = 2n,
q = 3n-3
p+q = 5n-3
\[
\left\lfloor \frac{p + q}{2} \right\rfloor = \left\lfloor \frac{5n - 3}{2} \right\rfloor \\
= 2n + \left\lfloor \frac{n - 3}{2} \right\rfloor \\
\geq 2n + 1 (\because n \geq 5)
\]

\[
\beta_0 + \beta_1 = \begin{cases} 
    n + n & \text{if } n \text{ is even} \\
    n + (n - 1) & \text{if } n \text{ is odd}
\end{cases}
\]

\[
\beta_0 + \beta_1 \leq 2n < 2n + 1 \leq \left\lfloor \frac{p + q}{2} \right\rfloor
\]

Hence \(D(P_n)\) is not highly total prime when \(n \geq 5\).

Highly total prime labeling of \(D(P_n)\) are given below:

**Highly Total Prime Labeling of \(D(P_2)\)**

**Highly Total Prime Labeling of \(D(P_3)\)**
Highly Total Prime Labeling of $D(P_4)$
4. CONCLUSION
In this paper we have discussed new type of graph labeling in detail. The discussion includes definition and results for highly total prime labeling techniques. Every highly total prime family of graphs gives rise to a prime family of graphs. Moving around in this area and mingling the different types of prime labeling cater to the need of researchers.

5. REFERENCES

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