Abstract

One of the purposes of System Reliability Analysis is to identify the weakness in a System and to quantify the Components failure. Unbiased Variance Estimates in System Reliability helps us to analyze the System based on the biased and unbiased nature of the System and Components. In this paper, we study Mixed Configuration System and evaluated its Variance Estimates of System Reliability Estimates. Numerical examples related to Medical Field are provided to illustrate the proposed idea in this paper.
### Notations

- $r_i$ = Reliability of $i^{th}$ Component, an unknown parameter
- $q_i$ = Unreliability of $i^{th}$ Component, $q_i = 1 - r_i$.
- $\hat{r}_i$ = Reliability Estimate of the $i^{th}$ Component
- $\hat{q}_i$ = Unreliability Estimate of the $i^{th}$ Component
- $\text{var}(\hat{r}_i)$ = Variance of Reliability Estimate of the $i^{th}$ Component
- $\text{var}(\hat{q}_i)$ = Variance of Unreliability Estimate of the $i^{th}$ Component
- $\text{var}(\hat{r}_i)$ = Variance Estimate of Reliability Estimate of the $i^{th}$ Component
- $\text{var}(\hat{q}_i)$ = Variance Estimate of Unreliability Estimate of the $i^{th}$ Component
- $\text{var}(\hat{r}_{ij})$ = Variance Estimate of Reliability Estimate of the $ij^{th}$ Component
- $\text{var}(\hat{q}_{ij})$ = Variance Estimate of Unreliability Estimate of the $ij^{th}$ Component
- $\text{var}(\hat{r}_{ijk})$ = Variance Estimate of Reliability Estimate of the $ijk^{th}$ Component
- $n_i$ = sample size in life testing for Component $i$
- $x_i$ = no. of survivals for $i^{th}$ Component life test
- $y_i$ = no. of failures for $i^{th}$ Component in life test
- $t$ = test duration for Components
- $\hat{R}_s$ = Reliability Estimate of Series System
- $\hat{R}_p$ = Reliability estimate of Parallel System
- $\text{var}(\hat{R}_{ss})$ = Variance Estimate of Reliability Estimate of Series-Series System
- $\text{var}(\hat{R}_{ps})$ = Variance Estimate of Reliability Estimate of Parallel-Series System
- $\text{var}(\hat{R}_{sss})$ = Variance Estimate of Reliability Estimate of Series-Series-Series System
1. Introduction

The Technological achievements of the last 50 years can hardly be disputed; there is one weakness in all mankind's devices. That is the possibility of failure. Each and every person has experienced the frustration of an automobile that fails to start or a malfunction of a household appliance. The introduction of every new device must be accompanied by provision for maintenance, repair parts, and protection against failure. The problem pervades modern society, from the homeowner who faces the annoyances of appliance failures, to electric utility companies faced with the potentially disastrous consequences of nuclear reactor failures [10].

Estimating Reliability is essentially a problem in Probability Modeling. A System consists of a number of components. In the simplest case, each Component has two states, operating or failed. When the set of operating Components and the set of failed Components are specified, it is possible to discern the status of the System. The problem is to compute the probability that the System is operating which determines Reliability of the System.

In Reliability Evaluation Studies, Engineering Systems may form various types of Configurations. If the Reliability factor or the Probability of failure of the System is to be determined, we will find that it is very difficult to analyze the System in its entirety. The failure of the System as a whole can be attributed to the failure of one or more Components of the System not functioning in the stipulated manner. In practice, the System is broken down into subsystems and the elements whose individual Reliability factors can be estimated or determined.

The development of Science and Technology and the needs of modern society are racing against each other. Industries are trying to introduce more and more automation in their industrial process in order to meet the ever increasing demands of Society. The complexity of Industrial Systems has therefore acquired special importance in recent years. Health Care Industry has a turnover of $145 billion annually and is currently expanding approximately at a rate of 7% annually. System Reliability approach to problems in medical field helps us to analyze the overall System Reliability of medical instruments and other such Systems in the medical field from the Reliability factors of the subsystem [7].

System Reliability approach to problems in Medical Field helps us to analyze the overall System Reliability of medical instruments and other such Systems in the medical field, from the Reliability factors of the subsystems and elements. The failure of the System as a whole can be attributed to the failure of one or more Components of the System not functioning in the stipulated manner. Reliability Engineering and Reliability Characteristics are playing an important role in almost all engineering disciplines [7].

System Reliability provides important knowledge for evaluating the new design and it is estimated based on individual Component Reliability. If these Component Reliability values are used to Estimate the System Reliability, the Estimation uncertainties of Component Reliability are propagated to System level. This often results in the Estimation uncertainty for the System Reliability Estimate [5].

The Variance is one of the most common risk measures used in the literature. The variance is a statistical metric which can be used to quantify the Estimation uncertainty of the Reliability Estimate. Estimating minimization of uncertainty are of particular importance for many decision makers to achieve higher Reliability and lower Reliability uncertainty because most decision makers prefer to obtain a risk-averse design [9].

In this paper we study Mixed Configuration System and its Variance Estimates. The paper is organized as follows. In Section 2 we give the basic definitions. In Section 3 we give the Variance
Estimates of System Reliability estimates of Mixed Series Configuration (1-step, 2-step, 3-step branching) Models. In Section 4 we analyze the Variance Estimates with an Illustrative numerical Example from Medical field.

2. Basic Definitions And Related Concepts

2.1: Series System

In a Series System, all Components in the System should be operating to maintain the required operation of the System. Thus, the failure of any one Component of the System will cause failure of the whole System.

Series configuration

![Series configuration](image)

Figure 1: Block diagram of K unit series system

The System Reliability Estimate of the Series System is given by

\[ R_s = \prod_{i=1}^{m} P(X_i) \]

where \( P(X_i) \) is the Reliability of the \( i^{th} \) component

2.2: Parallel System

In a Parallel System, the System operates if one or more Components operate, and the System fails if all the Components fail. The Parallel \( n \)-components are represented by the following block diagram.

Parallel configuration

![Parallel configuration](image)

Figure 2: \( n \) units parallel system

\[ R_p(t) = 1 - \prod_{i=1}^{n} (1 - P(X_i)) \]

where \( P(X_i) \) is the probability of the \( i^{th} \) component.
**Unbiased Estimate**

The Bias of an Estimator is defined to be

\[ \text{Bias} \left( \hat{\theta} \right) = E \left( \hat{\theta} \right) - \theta. \]

An estimate \( \hat{\theta} \) of parameter \( \theta \) is said to be Unbiased if its Bias is equal to zero for all values of parameter \( \theta \).

### 2.3 Fault Tree Analysis (FTA) approach

FTA is a top-down approach of a System analysis that is used to determine the possible occurrence of undesirable events or failures. Over the years, the method has gained favor over other reliability analysis approaches because of its versatility in degree of detail of complex systems. There are many symbols used to construct fault trees. The basic four symbols are

![Fault Tree Diagram](image)

**Figure 3**: A fault event that occurs from the logical combination of fault events through the input of logic gates such as OR and AND

![Fault Event](image)

**Figure 4**: A basic fault event

- Output event (Faults)
- OR gate

![Input Event](image)

**Figure 5**: The output fault event if one or more of input fault events occur.
Output event (faults)
AND gate

Input event (faults)

Figure 6: An output fault tree event occurs if all the input fault events occur.

3. Variance Estimation

Component Reliability Estimates and the Variance associated with these Estimates can be derived from various types of data including Binomial testing data. For the $i^{th}$ type of Component $(i=1,2...m)$ used in the System, let $n_i$ be the number of units on test for $t$ hours and let $x_i$ be the number of units survived during the testing period. Component survival number, a random variable, can be modeled by a Binomial distribution of $B(n_i, r_i)$. Then the Unbiased Estimate for $r_i$ and the exact Variance of the Estimate $\hat{r}_i$ are given by

$$\hat{r}_i = \frac{x_i}{n_i}$$ (1)

$$\text{var}\left(\hat{r}_i\right) = \frac{r_i(1-r_i)}{n_i}$$ (2)

The actual $r_i$ is not known in (2) and if the Estimate $\hat{r}_i$ is used to replace $r_i$, then an approximation of the Variance is given as

$$\text{var}\left(\hat{r}_i\right) = \frac{\hat{r}_i(1-\hat{r}_i)}{n_i}$$ (3)

Equation (3) is a Biased Estimate for $\text{var}\left(\hat{r}_i\right)$.

An Unbiased Estimate of $\text{var}\left(\hat{r}_i\right)$ is

$$\text{var}\left(\hat{r}_i\right) = \frac{\hat{r}_i(1-\hat{r}_i)}{n_i-1}$$

Series System: The variance of the Reliability Estimate for the Series System is given by

$$\text{var} \left( \hat{R}_s \right) = \text{var} \left( \prod_{i=1}^{m} \hat{r}_i \right) = \text{var} \left( \prod_{i=1}^{m} \frac{X_i}{n_i} \right)$$

where $X_i$ follows a binomial distribution $B(n, r)$ for component type $i = 1, 2, ... m$.

$$E \left( \prod_{i=1}^{m} X_i \right) = \prod_{i=1}^{m} E \left( X_i \right) = \prod_{i=1}^{m} \left( \frac{r_i}{n_i} \right)$$

which gives

$$\text{var} \left( \hat{R}_s \right) = \prod_{i=1}^{m} \left( r_i^2 + \text{var} \left( \hat{r}_i \right) \right) - \prod_{i=1}^{m} r_i^2$$

replace $r_i$ by $\hat{r}_i$ and $\text{var} \left( \hat{r}_i \right)$ by $\text{var} \left( \hat{r}_i \right)$ we get the biased estimate which is given as follows

$$\left[ \text{var} \left( \hat{R}_s \right) \right]_b = \prod_{i=1}^{m} \left( \hat{r}_i^2 + \text{var} \left( \hat{r}_i \right) \right) - \prod_{i=1}^{m} \hat{r}_i^2$$

An unbiased estimate for $\text{var} \left( \hat{R}_s \right)$ is given by

$$\left[ \text{var} \left( \hat{R}_s \right) \right]_{ub} = \prod_{i=1}^{m} \hat{r}_i^2 - \prod_{i=1}^{m} \left( \hat{r}_i^2 - \text{var} \left( \hat{r}_i \right) \right)$$

since

$$E \left( \text{var} \left( \hat{R}_s \right) \right) = \text{var} \left( \hat{R}_s \right)$$

Parallel System

Reliability Estimate for a Parallel System is given by $\hat{R}_p = 1 - \hat{Q}_p$ (9)

where $\hat{Q}_p = \prod_{i=1}^{m} q_i = \prod_{i=1}^{m} \left( \frac{1 - Y_i}{n_i} \right)$ for component i, assuming $n_i$ units are tested for time $t$ and $Y_i$ units failed. Let $Y_i = 1 - X_i$. Then

$$E \left( Y_i \right) = 1 - E \left( X_i \right) = 1 - \mu_i$$

$$\text{var} \left( Y_i \right) = \text{var} \left( X_i \right) = \sigma_i^2$$

$$\text{var} \left( \hat{R}_p \right) = \text{var} \left[ 1 - \left( \prod_{i=1}^{m} \frac{Y_i}{n_i} \right) \right] = \text{var} \left( \prod_{i=1}^{m} Y_i \right)$$
\[
E\left(\prod_{i=1}^{m} Y_i^2\right) - \left[E\left(\prod_{i=1}^{m} Y_i\right)\right]^2
\]
\[
\prod_{i=1}^{m} n_i^2
\]
\[
\prod_{i=1}^{m} E(Y_i^2) - \prod_{i=1}^{m} [E(Y_i)]^2
\]
\[
\prod_{i=1}^{m} n_i^2
\]
\[
= \prod_{i=1}^{m} \left[q_i^2 + \var\left(\hat{q}_i\right)\right] - \prod_{i=1}^{m} q_i^2
\]

But
\[
\var\left(\hat{Q}_p\right) = \var\left(1 - \hat{R}_p\right) = \var\left(\hat{R}_p\right)
\]
\[
\var\left(\hat{R}_p\right) = \prod_{i=1}^{m} \left[q_i^2 + \var\left(\hat{q}_i\right)\right] - \prod_{i=1}^{m} q_i^2.
\]

Replacing \( q_i \) by \( \hat{q}_i \) and \( \var\left(\hat{q}_i\right) \) by \( \var\left(\hat{q}_i\right) \), we obtain
\[
\var\left(\hat{Q}_p\right) = \prod_{i=1}^{m} \left[\hat{q}_i^2 + \var\left(\hat{q}_i\right)\right] - \prod_{i=1}^{m} \hat{q}_i^2
\]

which is a biased estimate for \( \var\left(\hat{Q}_p\right) \)
\[
\Rightarrow \var\left(\hat{R}_p\right) = \prod_{i=1}^{m} \left[1 - \hat{r}_i^2\right] + \var\left(1 - \hat{r}_i\right) - \prod_{i=1}^{m} \left(1 - \hat{r}_i\right)^2
\]
\[
= \prod_{i=1}^{m} \left[1 - \hat{r}_i^2\right] + \var\left(\hat{r}_i\right) - \prod_{i=1}^{m} \left(1 - \hat{r}_i\right)^2
\]

which is a biased estimate for \( \var\left(\hat{R}_p\right) \). The unbiased estimate for \( \var\left(\hat{R}_p\right) \) is
\[
\left[\var\left(\hat{R}_p\right)\right]_{ub} = \prod_{i=1}^{m} \left(1 - \hat{r}_i\right)^2 - \prod_{i=1}^{m} \left[1 - \hat{r}_i^2\right] - \var\left(\hat{r}_i\right)
\]
Since $E\left(\text{var} \left(\hat{R}_p\right)\right) = \text{var} \left(\hat{R}_p\right)$. The proof is similar to that of Series System.

### 4. Mixed Configuration Models (2-step branching)

#### i) Series - Series Model

Consider a System consisting of $i = 1, 2...m$ subunits connected in Series. Each of the subunits has $j = 1, 2...n$ subunits connected in Series, provided all the subunits are independent.

To find the Biased Variance Estimate of the Reliability Estimate, we replace $\hat{r}_j$ by $\prod_{j=1}^{n} \hat{r}_j$ in equation (6), we arrive

\[
\left[\text{Var} \left(\hat{R}_{ss}\right)\right]_{ab} = \prod_{i=1}^{m} \left[\prod_{j=1}^{n} \hat{r}_j\right]^2 + \text{Var} \left(\prod_{j=1}^{n} \hat{r}_j\right) - \prod_{i=1}^{m} \left(\prod_{j=1}^{n} \hat{r}_j\right)^2
\]  

(14)

Similarly the Unbiased Variance Estimate for a Mixed-Series System Reliability Estimate we replace $\hat{r}_j$ by $\prod_{j=1}^{n} \hat{r}_j$ in Equation (7)

\[
\left[\text{Var} \left(\hat{R}_{ss}\right)\right]_{ab} = \prod_{i=1}^{m} \left[\prod_{j=1}^{n} \hat{r}_j\right]^2 - \prod_{i=1}^{m} \left(\prod_{j=1}^{n} \hat{r}_j\right)^2 - \text{Var} \left(\prod_{j=1}^{n} \hat{r}_j\right)
\]  

(15)

#### ii) Parallel - Series model

To find the biased and unbiased variance Estimate for Parallel-Series Model we replace $\hat{r}_j$ by $\prod_{j=1}^{n} \hat{r}_j$ in Equation (12) and (13) respectively,

\[
\hat{\text{var}} \left(\hat{R}_{ps}\right) = \prod_{i=1}^{m} \left[1 - \prod_{j=1}^{n} \hat{r}_j\right]^2 + \text{var} \left(\prod_{j=1}^{n} \hat{r}_j\right) - \prod_{i=1}^{m} \left(1 - \prod_{j=1}^{n} \hat{r}_j\right)^2
\]  

(16)

\[
\hat{\text{var}} \left(\hat{R}_{ps}\right)_{ab} = \prod_{i=1}^{m} \left(1 - \prod_{j=1}^{n} \hat{r}_j\right)^2 - \prod_{i=1}^{m} \left(1 - \prod_{j=1}^{n} \hat{r}_j\right)^2 - \text{var} \left(\prod_{j=1}^{n} \hat{r}_j\right)
\]  

(17)

### 5. Mixed Configuration Models (3-step branching)

#### i) Series - Series -Series Model

Consider a System consisting of $i = 1, 2...m$ units connected in Series. Each of the unit has $j = 1, 2...n$ subunits connected in Series and each of the subunits have $k = 1, 2...r$ subunits connected in Series, provided all the subunits are independent.
To find the Biased Variance Estimate of the System Reliability Estimate, we replace $\hat{r}_{ij}^r$ by $\prod_{k=1}^{r} r_{ijk}^r$ in equation (14), we arrive

$$\left[ \text{Var} \left( \hat{R}_{sss} \right) \right]_{b} = \prod_{i=1}^{m} \left[ \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right]^2 + \text{Var} \left( \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right) - \prod_{i=1}^{m} \left[ \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right]^2$$

(18)

Similarly for the unbiased variance estimate for a Series-Series-Series System Reliability Estimate we replace $r_{ij}^r$ by $\prod_{k=1}^{r} r_{ijk}$ in equation (15) and we arrive

$$\left[ \text{Var} \left( \hat{R}_{sss} \right) \right]_{ub} = \prod_{i=1}^{m} \left[ \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right]^2 - \prod_{i=1}^{m} \left[ \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right]^2 - \text{Var} \left( \prod_{j=1}^{n} \left( \prod_{k=1}^{r} r_{ijk}^r \right) \right)$$

(19)

6. Illustrative Examples

Chemotherapy is used to treat many cancers. More than 100 Chemotherapy drugs are used today –either alone or in combination with other drugs or treatment. These drugs vary widely in their chemical composition, how they are taken, their usefulness in treating specific forms of cancer and their side effects. Knowing how the drug works is important. This helps the doctors to decide which drugs are likely to work well together.

Different doctors might choose different drug combinations with different schedules. This is typically more effective than a single drug, because the cancer cells can be affected in several different ways. Factors to consider in choosing which drugs to use include: The type of cancer, the stage of the cancer, patient’s age, health problems and types of cancer treatment in the past.

Doctors must calculate chemo doses were which precisely based on body weight in kilograms and body surface area (BSA) calculated using height and weight. Dosages for children and adults differ even after BSA is taken in to account. This is because Children’s bodies process drugs differently. They may have different levels of sensitivity to the drugs, too. The most effective doses and schedules of drugs to treat specific cancers have been found by testing them with samples.

7. PS Model

Consider the Blood cancer disease known as Acute Lymphoblastic Leukemia (ALL) which is most common leukemia among children between the ages 2 and 5. The diagnostic test which the pathologist conducts is Immunophenotyping. During this test, the pathologist looks at the specimen to see if the leukemia originates from B or T lymphocytes. Making this distinction helps physicians to recommend best treatment. About 85 percent of ALL derives B-lymphocytes known as Precursor B ALL. It is a low-risk type of ALL than ALL originating from T- lymphocytes. The treatment for ALL directs an injection of chemotherapy drugs such as Vincristine, Prednisolone, anthra-cyclines, L-Asparagine’s and other agents are given. During each phase there drugs are given to stop the growth of cancer cells throughout the body. The Fault tree Analysis given below predicts the Reliability of a child affected by ALL, where the treatment has been suggested with a combination of Chemotherapy drugs in a suitable composition according to the health status of the child.
Figure 7: The base event probabilities are calculated from a sample of size 30

8. SSS Model
Consider the problem of determining the Reliability of Insulin administered in correct dosage to a diabetic patient. The FTA diagram given below deals with Reliability of Insulin administered in correct dosage which depends on both correct sugar level measured and non-failure of the delivery system. For the sugar level to be correctly measured there should be no sensor failure or no error in sugar computation. The delivery System does not fail if Insulin computation is correct and pump signals correctly. Sensor failure does not occur if there is no sensor breakdown and data sensor is correct. The sugar computation error does not occur if the input blood parameters are correct (Algorithmic error) and analyzing correct blood sugar level (Arithmetic error). The Insulin computation error does not occur if the input blood sugar level is correct and computation algorithm...
is correct. The pump signals are correct due to correct input pump control commands and no breakdown of Insulin pump. Based on this analysis of the problem, the FTA diagram is given below.

Figure 8: All the events are independent and the base event probabilities are calculated from a small sample of size 30.
For this example, all the events are independent and the base event probabilities are calculated from a small sample of size 30.

Let us consider the values of the reliability estimates of the sub-basic events in the PS Model:

\[ \hat{r}_{11} = 0.3, \hat{r}_{12} = 0.2, \hat{r}_{13} = 0.3, \hat{r}_{14} = 0.1, \hat{r}_{15} = 0.4, \]
\[ \hat{r}_{21} = 0.5, \hat{r}_{22} = 0.4, \hat{r}_{23} = 0.3, \hat{r}_{24} = 0.1, \hat{r}_{25} = 0.2 \]

Let us consider the values of the reliability estimates of the sub-basic events for the SSS Model:

\[ \hat{r}_{111} = 0.667, \hat{r}_{112} = 0.5, \hat{r}_{121} = 0.333, \hat{r}_{122} = 0.83, \hat{r}_{211} = 0.533, \hat{r}_{212} = 0.467, \hat{r}_{221} = 0.6, \hat{r}_{222} = 0.4 \]

We consider the four types of study for the above examples:

- Biased System Biased Components (BSBC)
- Unbiased System Unbiased Components (USUC)
- Biased System Unbiased Components (BSUC)
- Unbiased System Biased Components (USBC)

### Table 1: Comparison

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<th>SSS Model</th>
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<tr>
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Figure 9: Comparision of PS Model

Figure 10: Comparision of SSS Model
10. Conclusion

In this Paper, we have derived the formulae to evaluate the biased and unbiased variance estimates of System Reliability for the mixed configuration (SSS and PS) models. Comparative study has been made based on the biased and unbiased nature of the System and Components. The Variance Estimate of System Reliability Estimate is more in Unbiased components of the System than biased components (i.e. Var of BSUC > Var of BSBC and Var of USUC > Var of USBC) whereas the Variance Estimate of System Reliability Estimate is less in unbiased System than in biased System (i.e. Var of USBC < Var of BSBC and the Var of USUC < Var of BSUC). Hence it is clear that the Variance Estimate of the System Reliability Estimate mainly depends on the unbiased nature of the Components, irrespective of the System being unbiased.

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References