Abstract
To make processing faster than human brain, we need to make use of machine (computer). Machine has got no intuition, no intelligence but circuits and devices; It can be fed with such data which it can process. Thus it can be made intelligent artificially; it can be made expert in any area say: medical diagnosis, chess game, washing a cloth, robot movement, etc., artificially. For this reason, a mathematical modelling of vague concept or vague knowledge or imprecise data or ill-defined information is necessary. An important tool, probably one of the most important tools, to do so for such information processing is fuzzy set theory. Among the various paradigmatic changes in science and mathematics in this century, one such changes the concept of uncertainty. In real world, has some special characterization by which it is possible to learn and reason in a vague and fuzzy environment. It has the ability to arrive at decision based on imprecise, qualitative data in contrast to formal mathematics and formal logic, which demand precise and quantitative data. The Probability theory has been an effective tool to handle uncertainty, but it can be applied to situations whose characteristics and based on random processes, i.e. the processes in which the occurrence of events is strictly determined by chance.

1. Introduction
It is generally agreed that an important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh, even though some ideas presented in the paper were envisioned some 30 years by the American philosopher Max Black. In this paper, Zadeh introduced a theory whose objects – fuzzy sets – are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of a degree.
Let us start with a remark that the purpose of a (conventional) set in mathematics is to formally characterize some concept. For instance, the “integer numbers which are greater than or equal to 3 and less than or equal to 10” may be uniquely represented by the following set:

\[
\{ x \in \mathbb{I} : 3 \leq x \leq 10 \}
\]

\[
\text{i.e. } \{ 3,4,5,6,7,8,9,10 \}
\]

where \( \mathbb{I} \) is the set of integers. However a serious difficulty occurs if we try to use the above notion of sets to characterize some less precise and somewhat vague concepts, e.g., the “integers which are more or less equal to 6”.

The source of vagueness is “more or less”. It is obvious that the conventional set is somewhat inadequate. Here; namely, it may be equated its characteristic function

\[
\psi : \mathbb{X} \rightarrow \{ 0, 1 \}
\]

which associates with each element of a discourse \( \mathbb{X} \) (in our case the set of integers \( \mathbb{I} \)) either 1 or 0 which means that a particular element either belongs to the set or does not, respectively. Thus, there is a clear distinction of the elements belonging and not belonging to the set, or equivalently the transition from ‘belonging’ to ‘not belonging’ to the set is abrupt, but for the “integers being more or less equal to 6”, such a precise distinction is evidently artificial. Here it is not possible to adequately fix any precise borderline.

The above line of reasoning suggested by L.A. Zadeh in 1965 was to replace the characteristic function \( \psi \) by the so called membership function defined as

\[
\mu : \mathbb{X} \rightarrow [0, 1]
\]

Thus the transition from ‘belonging’ to ‘not belonging’ to a set is now not clear-cut and abrupt, but gradual: from the full membership \( ( \mu(x) = 1 ) \) through intermediate membership \( ( 0 < \mu(x) < 1 ) \) to the full non-membership \( ( \mu(x) = 0 ) \). We have therefore some means for handling vagueness.

### 2 Some Useful Definitions

Suppose, \( \mathbb{X} \) is a non-null set, A fuzzy set A of this set \( \mathbb{X} \) is defined by the following set of pairs

\[
A = \{ (x, \mu_A(x)) : x \in \mathbb{X} \}
\]

where, \( \mu_A : \mathbb{X} \rightarrow [0, 1] \)

is a mapping called the membership function of A and \( \mu_A(x) \) is the grade of membership.

![Figure 2.1: Fuzzy set “Integers numbers more or less equal to 6”](image)
or degree of belongingness or degree of membership of \( x \in x \) in \( A \). Thus, a fuzzy set is a set of pair consisting of the particular elements of the universe of discourse and their membership grades. For brevity, however, we will often equate fuzzy sets with their membership functions, i.e. instead of “fuzzy set \( A \) characterized by \( \mu_A(x) \)” we will often say “fuzzy set \( \mu_A \)”.

The assumed set of values of the membership function \([0,1]\), may evidently be generalized for instance to some lattice, but such a case will not be considered in this thesis. For the purpose of this thesis work, \([0,1]\) is relevant enough from the conceptual viewpoint.

### 2.1 Basic Operation on Fuzzy Sets

As in the conventional (non-fuzzy) set theory, the basic operations in the theory of fuzzy sets are the complement, union and intersection. The following definitions of these operations were originally proposed by Zadeh and adopted by most of his followers. We will now define those operations. For brevity, the definitions will be given in terms of the respective membership functions.

#### 2.1.1 Complement of a fuzzy set

The Complement of a fuzzy set \( A \subseteq X \), write \( A^c \) is defined as

\[
\mu_{A^c}(x) = 1 - \mu_A(x) \quad \forall \ x \in X.
\]

**Example**

If \( x = \{1,2,3\} \) and \( A = 0.1/1 + 0.7/2 + 1/3 \), then \( A^c = 0.9/1 + 0.3/2 + 0/3 \)

The idea of complement is even more clear when presented graphically as in Figure-2 given below:

![Figure 2.2: Complement](image)

#### 2.1.2 Union

The union of a fuzzy set \( A, B \subseteq X \), write \( A \cup B \) is defined as

\[
\mu_{A \cup B}(x) = \mu_A(x) \lor \mu_B(x) \quad \forall \ x \in X.
\]

Where “\( \lor \)” is the ‘maximum’ operator.

**Example**

If \( x = \{1,2,3,4\} \), \( A = \frac{2}{1} + \frac{5}{2} + \frac{8}{3} + \frac{1}{4} \), \( B = \frac{7}{1} + \frac{8}{2} + \frac{5}{3} + \frac{2}{4} \) then

\[
A \cup B = \frac{1}{1} + \frac{8}{2} + \frac{8}{3} + \frac{1}{4}.
\]
The essence of union may well be seen in Figure.

\[
\text{Figure 2.3: Union}
\]

2.1.3 Intersection

The Intersection of a fuzzy set \( A, B \subseteq X \), write \( A \cap B \) is defined as

\[
\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) \quad \forall x \in X,
\]

Where “\( \wedge \)” is the ‘minimum’ operator.

\[
\text{Figure 2.4: Intersection}
\]

Example

If \( x = \{1, 2\}, A = \frac{1}{1} + \frac{7}{2}, B = \frac{5}{1} + \frac{9}{2} \) then \( A \cap B = \frac{5}{1} + \frac{7}{2} \).

The essence of intersection may clearly be seen in above figure.

2.1.4 Fuzziness and randomness

Fuzziness and randomness is some sort of uncertainty, but they are different concepts. The randomness deals with well-defined phenomena (events) whose occurrence is uncertain. The fuzziness concerns concepts and entities which are not crisply defined.

And what about a joint occurrence of fuzziness and randomness? Such situations are quite common, e.g., when we ask about the probability of “good” weather tomorrow, of “much lower” inflation, etc.
2.2 Image And Inverse-Image

2.2.1 Inverse image of a fuzzy sets

Let X and Y be two sets and f be a mapping from X into Y. Let B be a fuzzy set in Y. Then the inverse image $f^{-1}[B]$ of B is the fuzzy set in X with the membership function defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \quad \forall x \in X$$

2.2.2 Image of a fuzzy sets

Also let A be a fuzzy set in X. Then the image of A, $f[A]$ is the fuzzy set in Y with membership function defined by

$$\mu_{f[A]}(y) = \sup_{z \in f^{-1}(y)} \mu_A(z) \quad \text{if } f^{-1}(y) \neq \emptyset$$

$$= 0, \text{otherwise.}$$

$\forall y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$.

3 Fuzzy Sets

In 1975 Zadeh made an extension of the concept of a fuzzy set by an interval valued fuzzy set (i.e. a fuzzy set with an interval valued membership function). The interval-valued fuzzy set is referred to as i-v fuzzy set. Zadeh constructed a method of approximate inference using this i-v fuzzy set. Subsequently Gorzalezcyz studied the i-v fuzzy sets for approximate reasoning. Fuzzy relations have been studied and applied by many authors in several directions like pattern recognitions, character recognitions. Sanchez used fuzzy relations in medical diagnosis. In this paper the present author defines and discusses i-v fuzzy relations using interval valued membership function the paper begins with a brief review of earlier research works on i-v fuzzy sets, some algebraic operations on those, fuzzy relations, sup-min composition. Then the author introduced the concepts of i-v fuzzy relations, i-v fuzzy tolerance relations, i-v fuzzy equivalence relations and composition of i-v fuzzy relations. Several useful propositions are also proved. Finally the pioneering approach of Sanchez in medical diagnosis is considered and discussed in the light of i-v relations. It is claimed that use of i-v fuzzy relations instead of point valued fuzzy relations sometimes leads to more realistic use of this technique of medical diagnosis.

3.1 i-v fuzzy sets

The formal fuzzy set representation, especially of verbal expression occurring in a verbal model of a phenomenon, object or process (in a verbal decision procedure or in case of methods of approximate inference) is not often sufficiently adequate. Methods of approximate inference based on fuzzy set theory allow formal representation to be built for verbal decision-making procedures containing vague, fuzzy premises. In experimental investigations, fuzzy inference algorithms almost always assume the form of computer algorithms. That is why both (1) formal mathematical properties of fuzzy methods of approximate inference and (2) the possibilities of their effective computer representation belong to important factors determining the final outcome of applying these methods.
The former factor determines the ‘accuracy’ of the representation of a verbal decision procedure by a fuzzy algorithm.

The latter factor determines the accuracy of this algorithm’s representation in a computer, computation time and memory loading. Of course, the problem of formulations a verbal decision procedure is a separate issue which goes beyond the frames of fuzzy set theory. As a rule, the membership functions of fuzzy sets representing particular verbal expression can not be defined unequivocally on the basis of available information. Therefore, it is not always possible for a membership function of the type

\[ \mu : x \rightarrow [0,1] \]

to assign precisely one point from the interval \([0,1]\) to each element \(X \in X\) without the loss of at least a part of information. It should be emphasized, however, that from the point of view of practice, this type of formal representation of a generalized membership function is characterized by some ‘redundancy’ in relations to reality.

![Figure 3.1: Example of an i-v fuzzy set](image)

Referring to earlier considerations, it seems that the idea of an i-v fuzzy set, from the point a view of practice, is a sufficiently complete generalization of the concept of a fuzzy set. On the other hand it is not such a considerably negative influence on the effectiveness of the method of approximate inference based on i-v fuzzy sets.

In the light of the above consideration, a proposal is put forward to apply the extension of the concept of fuzzy set represented by generalized membership function having the form of ‘band’.

### 3.2 Union of two i-v fuzzy sets

If \(A\) and \(B\) are two i-v fuzzy sets, then their union is an i-v fuzzy set \(A \cup B \in IVF (X)\) given by (Figure):

\[
A \cup B = \{(x, \mu_{A\cup B}(x))\}, \text{ where}
\]

\[
\mu_{A\cup B}(x) = [\mu_{A\cup B}^L(x), \mu_{A\cup B}^U(x)], \quad x \in X.
\]

Where,
This definition may be generalized, in a natural way, to the case of a union of n i-v fuzzy sets.

3.3 Intersection of two i-v fuzzy sets

If A and B are two i-v fuzzy sets, then their union is an i-v fuzzy set A ∪ B ∈ IVF (X) given by (Figure):

\[ A \cap B = \{ (x, \bar{\mu}_{A \cap B} (x)) \} \]

\[ \bar{\mu}_{A \cap B} (x) = \left[ \mu_{A \cap B}^L (x), \mu_{A \cap B}^U (x) \right], x \in X. \]

Where,

\[ \mu_{A \cup B}^L (x) = \min \left[ \mu_{A \cap B}^L (x), \mu_{B}^L (x) \right]. \]

\[ \mu_{A \cup B}^U (x) = \min \left[ \mu_{A \cap B}^U (x), \mu_{B}^U (x) \right]. \]

This definition also may be generalized.

![Figure 3.2: Example of an i-v fuzzy set](image)

4 Fuzzy Relations

Fuzzy relation gives a measure of relation between two fuzzy sets in the range 0 to 1. Crisp sense, either relation exists (1) or does not (0). In fuzzy sense there exists always some relation between any two objects in the real world. That is why fuzzy relations have been a major topic in many areas like Medical Diagnoses, Management decision makings, Expert system, Image Processing, Economics etc., to list a few only.

4.1 Fuzzy Relation

Let x, y be two sets. A fuzzy relation R from X to y is a fuzzy subset of X x Y, characterized by its membership function:-
\( \mu_R : X \times Y \rightarrow [0,1] \)

A fuzzy relation \( R \) from \( Z \) to \( Y \) will be referred to as \( R (X \rightarrow Y) \).

**Example:** Say, \( X = \{2, 5\} \), \( Y = \{1, 9, 7\} \)

<table>
<thead>
<tr>
<th>( x \rightarrow y )</th>
<th>1</th>
<th>9</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>5</td>
<td>.7</td>
<td>.5</td>
<td>.9</td>
</tr>
</tbody>
</table>

4.2 **Sup-min composition of a relation with a Fuzzy Set**

When \( A \) is a fuzzy subset of \( X \), the sup-min composition of \( R \) and \( A \) is fuzzy subset \( B \) of \( Y \) defined by

\[
\mu_B (y) \text{ i.e. } \mu_{R \circ A} (y) = \sup_{x \in X} \min \{ \mu_A (x), \mu_R (x, y) \} \quad \forall y \in Y.
\]

which is usually rewritten with the \( \vee (\sup) \), and \( ^\wedge (\min) \) operators, deleting ‘\( \in X \)’ when no ambiguity arises, as below:

\[
\mu_{R \circ A} (y) = x \{ \mu_A (x) \vee \mu_R (x, y) \} \quad \forall y \in Y.
\]

4.3 **Sup-min composition of two Fuzzy Relations**

Let \( Q (x \rightarrow Y) \) and \( R (Y \rightarrow Z) \). The sup-min composition \( R \circ Q \) is a fuzzy relation from \( x \) to \( z \), defined by

Example: If \( x = \{1,2\} \), \( Y = \{1,2,3,4\} \) and \( Z = \{1,2,3,4\} \), then the composition of two fuzzy relations, \( R \) and \( S \), given below in the matrix form is

\[
\begin{array}{ccc}
R & o & S \\
\end{array}
\begin{array}{ccc}
1 & 2 & 3 \\
1.4 & 1.8 & 1.3 \\
2 & 4 & .7 & .5 \\
\end{array}
\begin{array}{ccc}
1 & 2 & 3 & 4 \\
1.7 & .6 & .4 & .1 \\
2 & .4 & .1 & .7 & .2 \\
3.5 & .9 & .6 & .8 \\
\end{array}
\begin{array}{ccc}
1 & 2 & 3 & 4 \\
1.5 & .9 & .7 & .8 \\
2 & .7 & .7 & .4 \\
\end{array}
\]

4.4 **i-v FUZZY RELATION**

An i-v fuzzy relation \( R \) is an i-v fuzzy set in a cartesian product \( x \times Y \) of universe \( X \) and \( Y \). The membership value of \((x,y)\) i.e. \( \mu_R (x,y) \) in \( R \) estimates the interval of the strength of the link between \( x \) and \( y \).

\[
\mu_R^\cup (x,y) \quad \text{and} \quad \mu_R^L (x,y)
\]

are respectively called the maximum strength and minimum strength of the link between \( x \) and \( y \). Two i-v fuzzy relations \( R_1 \) and \( R_2 \) are equal if
\forall (x, y) \in X \times Y,
\mu_{R_1}(\xi, \psi) = \mu_{R_2}(x, y).

4.5 Composition

Let A be an i-v fuzzy set of x and R be an i-v fuzzy relation from x to y. Then composition of R with A is B = R o A, which is an i-v fuzzy set of Y defined by

\[ \mu_{R o A}(y) = \left[ \text{Infimum}_{x \in X} \left( \mu_{A}^I(x) \wedge \mu_{R}^I(y, z) \right), \text{Supremum}_{x \in X} \left( \mu_{A}^{II}(x) \wedge \mu_{R}^{II}(y, z) \right) \right] \]

Let R and S be two i-v fuzzy relations on X x Y and Y x Z respectively. Then composition relation R o S is an i-v fuzzy set in X x Z with membership function defined by :

\[ \mu_{R o S}(x, z) = \left[ \text{Inf}_{x \in Y} \left( \mu_{R}^I(x, y) \wedge \mu_{S}^I(y, z) \right), \text{Sup}_{y \in Y} \left( \mu_{R}^{II}(x, y) \wedge \mu_{S}^{II}(y, z) \right) \right] \]

Thus an interval valued link between x and z is established through Y. This R o S is known as a composition of i-v fuzzy relations which we shall refer to henceforth as Civ FR.

An i-v fuzzy relation R on X x X is said to be

- **Reflexive**: iff for all x \in X, \( \mu_{R}^I(x, x) = [1, 1] \).
- **Symmetrical**: iff for all \( x_1, x_2 \in X \),
\[ \mu_{R}^I(x_1, x_2) = \mu_{R}^I(x_2, x_1) \]

4.6 Inverse of an i-v fuzzy relation

If R is an i-v fuzzy relation on X x Y, it’s inverse \( R^{-1} \) is an i-v fuzzy relation on Y x X such that
\[ \mu_{R^{-1}}(y, x) = \mu_{R}^I(x, y) \]

Clearly,
\[ \mu_{R^{-1}}^I(y, x) = \mu_{R}^I(x, y) \text{ and } \mu_{R^{-1}}^{II}(y, x) = \mu_{R}^{II}(x, y) \]

**Proposition**

(i) \((R^{-1})^{-1} = R\)
(ii) \((R o S)^{-1} = S^{-1} o R^{-1}\)

where R and S are any two i-v fuzzy relations on X x Y and Y x Z respectively.
(i) \( \tilde{\mu}_R^{-1}(x, y) = \mu_R^{-1}(y, z) \mu_S(x, y) \) \\
\( \forall x \in X, y \in Y. \) \\
\( \Rightarrow (R^{-1})^{-1} = R. \text{ Proved.} \)

(ii) \( R : X \rightarrow Y, S : Y \rightarrow Z, \) \\
\( \Rightarrow R \circ S : X \rightarrow Z, R^{-1} : Y \rightarrow X, \) \\
and \( S^{-1} : Z \rightarrow Y. \) \\
\( \Rightarrow (R \circ S)^{-1} : Z \rightarrow X, \) \( S^{-1} \circ R^{-1} : Z \rightarrow X. \)

Now, \( \tilde{\mu}_{R \circ S}^{-1}(z, x) \)
\[ = \tilde{\mu}_{R \circ S}^{-1}(x, z) \]
\[ = \left[ \inf_y (\mu_S^{-1}(x, y) \land \mu_R^{-1}(y, z)), \sup_y (\mu_S^{-1}(x, y) \land \mu_R^{-1}(y, z)) \right] \]
\[ = \left[ \inf_y (\mu_R^{-1}(x, y) \land \mu_S^{-1}(y, z)), \sup_y (\mu_R^{-1}(x, y) \land \mu_S^{-1}(y, z)) \right] \]
\[ = \mu_{S^{-1} \circ R^{-1}}(z, x) \]

5 Objectives of the study

The standard fuzzy operation are generalization of the corresponding classical set operation, as the standard fuzzy operations perform precisely as the corresponding operation for crisp sets when the range of membership grades is restricted to the set \( \{0,1\} \). Possibility theory is closely concerned with fuzzy set theory and plays an important role in some of its applications. Fuzzy set theory, fuzzy measures theory, probability theory and evidence theory, all of these theories are related and are used to characterize the various form of uncertainty.

The concept of information is intimately connected with the concept of uncertainty. The basic aspect of this connection is that uncertainty involved in any problem solving situation is a result of some deficiency in the information. The deficiencies in the information may be in the form of that it may be incomplete imprecise, fragmentary, not fully reliable, vague, contradictory or deficient in some other way. Thus, in general these various types of uncertainties.

The concept of information has been studied to measure the amount of uncertainty in a problem-solving situation conceptualized in a particular mathematical theory, information in human communication and cognition. The concept of information has also been studied in terms of the theory of computability, independent of the concept of uncertainty.

Making decisions is undoubtedly one of the most fundamental activities of human being. Thus, the subject of decision making is concern about the study of how decisions are actually made and how they can be made better or more successfully. Decision making is broadly defined to include any choice or selection of alternatives. Thus, it is therefore of importance in many fields in both the “soft” social science and the “hard” disciplines of natural science and engineering.
6 Conclusions
The concept of fuzzy set defined above makes it possible to develop a consistent generalized set theory, called the theory of fuzzy sets. During the past two decades engineering journals as well as those in the social science, Computer, Engineering, Economics, Information science, Mathematics, Medical Science etc. to name a few, have develop increasing space to article dealing with the analysis and applications of fuzzy set theory. Many scientists, mathematicians, social scientists, engineers, technologists have contributed to this progress but the literature on such a diversified subject is widely scattered.

References