Pseudo Topogenic And Totally Null Topogenic Graphs

Abstract

A new concept namely Pseudo Topogenic Graphs is introduced. Common families of graphs such as Path, Star and Complete Graphs are studied. Interrelations with the existing concepts namely Topogenic graphs has been analysed. Labeling the vertices and edges with subsets of a set X so that the label of the edges incident at a vertex are disjoint give rise to another new concept known as Totally Null Topogenic Graph and is discussed.

1. INTRODUCTION

The concept of Topogenic Graphs was introduced by B.D.Acharya et.al [1]. In topogenic graphs they have labeled the vertices with subsets of a ground set and edges by the symmetric difference of the label of vertices so that the labels form a topology. They have proved that the complete graph K4 is not topogenic. Labeling the vertices of K4 without the label \( \emptyset \), \( f(V) \cup f(\emptyset)(E) \cup \emptyset \) form a topology with ground set having 3 elements. This resulted in the first variation of Topogenic graphs namely Pseudo Topogenic graphs. Edge topogenic graphs dealt with the labeling of edges with subsets of X and vertices by the union of all the label of edges incident at
that vertex was introduced by R. B. Gnana Jothi and A. Uma Devi [2]. This motivated us to define a new term totally null topogenic graphs which is the labeling of \(|V \cup E|\) in the set-indexer.

2. PSEUDO TOPOGENIC GRAPHS

2.1 Definition
A graph \( G=(V,E) \) is said to be Pseudo Topogenic if it admits a set-indexer \( f : V \rightarrow 2^X \setminus \{ \varnothing \} \) such that \( f(V) \cup f^\oplus(E) \cup \varnothing \) is a topology on \( X \) where \( f^\oplus(uv) = f(u) \oplus f(v) \) for all \( uv \in E(G) \).

Example: Path \( P_3 \) is pseudo topogenic. Take \( X = \{1, 2, 3\} \)

\[
\begin{align*}
\{1\} & \quad \{2\} & \quad \{3\} \\
\{\}\ & \quad X & \quad \{1, 2\} & \quad \{1, 3\} & \quad \{1, 2, 3\}
\end{align*}
\]

Figure 2.1: Pseudo topogenic set-indexer of \( P_3 \)

\[
f(V) \cup f^\oplus(E) \cup \varnothing = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, X\}
\]

2.2 Definition
Pseudo topogenic index of a graph \( G \) is the minimum cardinality of ground set \( X \) such that \( G \) has a pseudo topogenic set-indexer with respect to \( X \).

2.3 Notation
Let \( Pti(G) \) denote the pseudo topogenic index of \( G \).

2.4 Observation
In a graph topogenic index may be strictly greater than pseudo topogenic index.

Example: Consider the path \( P_4 \).

For a topogenic set-indexer of \( P_4 \), we need at least 3 elements in \( X \). As there is no topology with 7 subsets of \( X=\{1,2,3\} \), we take 4 elements in \( X \). Topogenic set-indexer of \( P_4 \) is

\[
\begin{align*}
X & \quad \{3,4\} & \quad \{3\} \\
\{1,2\} & \quad X & \quad \{1,2,3\}
\end{align*}
\]

Figure 2.2: Topogenic set-indexer of \( P_4 \)

\[
\tau = \{\varnothing, X, \{3\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}\}
\]

\[\gamma(P_4) = 4.\]

For a pseudo topogenic set-indexer of \( P_4 \), 3 elements are needed in \( X \), since \( \varnothing \) cannot be labeled to any vertex. Take \( X=\{1,2,3\} \). Pseudo topogenic set-indexer of \( P_4 \) is
Figure 2.3: Pseudo topogenic set-indexer of P₄

\[ f(V) \cup f^{\oplus}(E) \cup \varnothing = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X \} \]

\[ P_{ti}(P_4) = 3 \]

Hence \( \Upsilon(P_4) > P_{ti}(P_4) \)

2.5 Observation
In a graph topogenic index may be equal to pseudo topogenic index.

**Example:** Consider the path P₃
As there are 3 vertices, at least 2 elements are needed in X. Topogenic set-indexer of P₃ with \( X = \{1, 2\} \)

Figure 2.4: Topogenic set-indexer of P₃

\[ \tau = \{\varnothing, \{1\}, \{2\}, X \}, \Upsilon(P_3) = 2 \]

Pseudo topogenic set-indexer of P₃ with \( X = \{1, 2\} \)

Figure 2.5: Pseudo topogenic set-indexer of P₃

\[ f(V) \cup f^{\oplus}(E) \cup \varnothing = \{\varnothing, \{1\}, \{2\}, X \}, P_{ti}(P_3) = 2 \]

Hence \( \Upsilon(P_3) = P_{ti}(P_3) \)

2.6 Observation
In a graph topogenic index may be strictly less than pseudo topogenic index.

**Example:** Consider the star graph \( K_{1,3} \).
Let \( X = \{1, 2\} \). It has 4 subsets. Assign \( \varnothing \) to the central vertex, remaining 3 subsets are assigned to the other vertices. Topogenic set-indexer of \( K_{1,3} \) is

Figure 2.6: Topogenic set-indexer of \( K_{1,3} \)

\[ \tau = \{\varnothing, \{1\}, \{2\}, X \}, \Upsilon(K_{1,3}) = 2 \]

For a Pseudo topogenic set-indexer, as \( \varnothing \) is not included for labeling the four vertices, at least 4 non-empty subsets of \( X \) are needed which is possible with \( X = \{1, 2, 3\} \). Assign \( \{1\} \) to the central vertex. Label the remaining vertices as \( f(v_i) = \{2, 3, .., i\}; \ 2 \leq i \leq n - 1 \) and \( f(v_n) = X \). Pseudo topogenic set-indexer of \( K_{1,3} \) is
2.7 Theorem

Star graph \((K_{1,n})\) is pseudo topogenic for every positive integer \(n\).

**Proof:**

Let \(V(K_{1,n}) = \{u_0, v_1, v_2, \ldots, v_n\}\) and
\[E(K_{1,n}) = \{u_0v_i/i = 1, 2, \ldots, n\}\]

Take the ground set \(X = \{1, 2, \ldots, (n + 1)\}\).
Label the vertices as \(f(u_0) = \{1\}, f(v_i) = \{1, 2, \ldots, (i + 1)\}; 1 \leq i \leq n\)

Then \(f^{\oplus}(u_0v_i) = \{2, 3, \ldots, (i + 1)\}; 2 \leq i \leq n\) and \(f(u_0) \subset f(v_1) \subset \ldots \subset f(v_{n-1}) \subset f(v_n)\)

\(f(u_0) \cup f(v_i) = f(v_i); 1 \leq i \leq n\)
\(f(u_0) \cap f(v_i) = \emptyset\)

Then \(f(V) \cup f^{\oplus}(E) \cup \emptyset\) forms a topology on \(X\). Hence Star graph is pseudo topogenic.

2.8 Note

Pseudo topogenic labeling need not be unique.

For example, another pseudo topogenic labeling of \((K_{1,n})\) with \(X = \{1, 2, \ldots, n\}\)
2.9 Theorem
The complete graph \((K_n)\) is pseudo topogenic for \(n = 3, 4, 5\).

**Proof:**
When \(n=3\), Take \(X=\{1,2\}\)

**Figure 2.10:** Pseudo topogenic set-indexer of \(K_3\)

\[f(V) \cup f(\oplus)(E) \cup \varnothing = \{\varnothing, \{1\}, \{2\}, X\}.\]

When \(n=4\), Take \(X=\{1,2,3\}\)

**Figure 2.11:** Pseudo topogenic set-indexer of \(K_5\)

\[f(V) \cup f(\oplus)(E) \cup \varnothing = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.\]

When \(n=5\), Take \(X=\{1,2,3,4\}\)

Hence complete graph \((K_n)\) is pseudo topogenic for \(n= 3, 4, 5\).

2.10 Note
Complete graphs \(K_4\) and \(K_5\) are not topogenic [3], but they are Pseudo topogenic.

2.11 Theorem.
\(P_t(K_n) = n - 1\), \(n = 3, 4, 5\)

**Proof:**
Theorem 2.9 provides Pseudo topogenic index for \(K_n\) with ground set
\(X= \{1,2,\ldots, (n-1)\}\)
\(|E(K_n)| = n^2 > 2^{n/2}\) for \(n=3,4,5\)
3. TOTALLY NULL TOPOGENIC GRAPHS

3.1 Definition
A graph \( G = (V, E) \) is said to be totally null topogenic if it admits a set-indexer \( f : V \cup E \rightarrow 2^X \) such that

(i) Range of \( f \) is a topology on \( X \).

(ii) \( \bigcap_{v \in N(u)} f(uv) = \emptyset \) \( \forall u \in V(G) \) with \( d(u) \geq 2 \)

(iii) \( 2^{|X|} - 1 < |V \cup E| \leq 2^{|X|} \)

**Example:** Cycle \( C_3 \) is totally null topogenic.

When \( n=3 \), Take \( X = \{1, 2, 3\}; 2^{|X|} = 8, 4 < |V \cup E| = 6 < 2^{|X|} \)

![Figure 3.1: Totally null topogenic set-indexer of \( C_3 \)](image)

\[ \tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X\} \]

3.1 Observation
In a connected graph \( X \) cannot be assigned to any edge in any totally null topogenic labeling.

3.3 Theorem
Any graph \( G \) with \( 3*2^{r-2} < |V \cup E| < 2^r \), is not totally null topogenic for some \( r \).

**Proof:**
Let \( G \) be a graph with \( 3*2^{r-2} < |V \cup E| < 2^r \), for some \( r \).

If \( G \) is totally null topogenic, the ground set \( X \) should be chosen such that \( 2^{|X|} - 1 < |V \cup E| < 2^{|X|} \) and \( f (V \cup E) \) is a topology on \( |V \cup E| \) elements.

Now, \( 2^{r-1} = 2*2^{r-2} < 3*2^{r-2} < 4*2^{r-2} = 2^r \).

By theorem **"For \( n \geq 3, T(n, k) = 0 \) for every \( k \) with \( 3*2^{n-2} < k < 2^n \) where \( T(n, k) \) is the set of all labeled topologies on \( X \) and having \( k \) open sets, \( 2 \leq k \leq 2^n \) [1], there is no topology with \( k \) elements for \( 3*2^{r-1} < k < 2^r \).**

Therefore, \( G \) is not totally null topogenic if \( 3*2^{r-2} < |V \cup E| < 2^r \), for some \( r \).
Corollary:
K₅ and K₇ are not totally null topogenic graphs.

Proof:
When r=4, 3*2⁴−2=12 and 2⁴=16.
In K₅ as |V U E|= 15, it is not totally null topogenic graph. When r=5, 3*2⁵−2
= 24 and 2⁵=32.
In K₇ as |V U E|= 28, it is not totally null topogenic.

3.4 Theorem
Path (Pₙ) is totally null topogenic for n = 3, 5, 9.

Proof:
When n=3 , Take X={1,2,3} ; 2^{|X|} = 8, 4 < |V U E | = 5 < 2^{|X|}.

\[ \tau = \{ \emptyset, \{1\}, \{2\}, \{1, 2\}, X \}. \]

When n=5 , Take X={1,2,3,4} ; 2^{|X|} = 16, 8 < |V U E | = 9 < 2^{|X|}.

\[ \tau = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X \}. \]

When n=9 ,Take X={1,2,3,4,5}, 2^{|X|} = 32, 16 < |V U E | = 17 < 2^{|X|}.

\[ \tau = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, X \}. \]

3.5 Theorem
Cycles Cₙ are totally null topogenic for 3 ≤ n ≤ 6, 8 ≤ n ≤ 10.

Proof:
When n=3 , Take X={1,2,3} ; 2^{|X|} = 8, 4 < |V U E | = 6 < 2^{|X|}.
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Figure 3.5: Totally null topogenic set-indexer of C₃
\[ \tau = \{ \phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, X \} \]
When \( n=4 \), take \( X = \{1,2,3\} \):
\[ 2|X| = 8, \quad 4 \leq |V \cup E| = 8 = 2|X| \]

Figure 3.6: Totally null topogenic set-indexer of C₄
\[ \tau = \{ \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X \} \]
We are able to accomplish till C₁₀. The results are tabulated below:

<table>
<thead>
<tr>
<th>Size of Cycle</th>
<th>Ground set</th>
<th>Indexer of f</th>
<th>f(( V \cup E ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( X = {1, 2, 3, 4} )</td>
<td>( f(e₁) = \phi,\ f(e₂) = {1},\ f(e₃) = {2},\ f(e₄) = {1, 3},\ f(e₅) = {2, 3} )</td>
<td>( {\phi, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, {2, 3, 4}, X } )</td>
</tr>
<tr>
<td>6</td>
<td>( X = {1, 2, 3, 4} )</td>
<td>( f(e₁) = \phi,\ f(e₂) = {1},\ f(e₃) = {2, 3},\ f(e₄) = {1, 4},\ f(e₅) = {2},\ f(e₆) = {4} )</td>
<td>( {\phi, {1}, {2}, {4}, {1, 2}, {1, 4}, {2, 3}, {2, 4}, {1, 2, 3}, {1, 2, 4}, {2, 3, 4}, X } )</td>
</tr>
<tr>
<td>8</td>
<td>( X = {1, 2, 3, 4} )</td>
<td>( f(e₁) = \phi,\ f(e₂) = {4},\ f(e₃) = {1, 2},\ f(e₄) = {3},\ f(e₅) = {1, 4},\ f(e₆) = {2},\ f(e₇) = {3, 4},\ f(e₈) = {1} )</td>
<td>( {\phi, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, X } )</td>
</tr>
<tr>
<td>9</td>
<td>( X = {1, 2, 3, 4, 5} )</td>
<td>( f(e₁) = \phi,\ f(e₂) = {1, 2, 4},\ f(e₃) = {3},\ f(e₄) = {4},\ f(e₅) = {2, 3},\ f(e₆) = {4, 5},\ f(e₇) = {2},\ f(e₈) = {3, 4, 5},\ f(e₉) = {1, 2} )</td>
<td>( {\phi, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4, 5}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4, 5}, X } )</td>
</tr>
<tr>
<td>10</td>
<td>( X = {1, 2, 3, 4, 5} )</td>
<td>( f(e₁) = \phi,\ f(e₂) = {4, 5},\ f(e₃) = {2},\ f(e₄) = {4},\ f(e₅) = {1, 2, 3},\ f(e₆) = {5},\ f(e₇) = {2, 4},\ f(e₈) = {3},\ f(e₉) = {2, 5},\ f(e₁₀) = {3, 4} )</td>
<td>( {\phi, {2}, {3}, {4}, {5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {4, 5}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5} } )</td>
</tr>
</tbody>
</table>
3.6 Theorem
Star graph \((K_{1,n})\) is totally null topogenic for \(n = 2, 4\).

Proof:
When \(n = 2\), take \(X = \{1, 2, 3\}; 2^{|X|} = 8\), \(4 < \mid V \cup E \mid = 5 < 2^{|X|}\).

\[
\text{Figure 3.7: Totally null topogenic set-indexer of } K_{1,2}
\]
\[\tau = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\} \]

When \(n = 4\), take \(X = \{1, 2, 3, 4\}; 2^{|X|} = 16\), \(8 < \mid V \cup E \mid = 9 < 2^{|X|}\).

\[
\text{Figure 3.8: Totally null topogenic set-indexer of } K_{1,4}
\]
\[\tau = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, X\}.
\]

4. CONCLUSION
Introduction of two variations on topogenic set-indexers give rise to so many problems. It is quite interesting to see that a slight variation brings some important graphs such as complete graphs to the arena of consideration. Giving equal importance to vertices and edges is practised in totally null topogenic graphs. Here again only elementary families are dealt with. Problems which arose during the period of working are listed below.

i.) Determination of necessary/ sufficient/ necessary and sufficient condition for a graph to be topogenic/ pseudo topogenic/ totally null topogenic.

ii.) Determining topogenic/ pseudo topogenic set index for families of graphs.

iii.) Obtaining bounds for pseudo topogenic set index for some families of graphs.

iv.) Determining totally null topogenic set index for complete graphs.

v.) With positive vibration that this is an useful piece for research, this article is concluded.

5. REFERENCES

6. BIOGRAPHIES

Dr. S.M.Meena Rani is working as an Associate Professor in Mathematics in V.V.Vanniaperumal College for Women, Virudhungar. She is having 30 years of teaching experience and 7 years of research experience. She has published more than 7 papers in international research journals. Her field of research is Graph Theory and Neural Network.

B.Eswari is a M.Phil Scholar in the department of Mathematics, V.V.Vanniaperumal College for Women, Virudhungar. Her field of research is Graph Theory.