Inverse domination number has been determined for some families of graphs such as Andrasfai graphs, Prime Complete graphs Book graphs, Shadow graphs, Windmill graphs, Two cycles with common edge and platonic solids. A biproduct of this concept is called as γγ-sets are also considered and some families of graphs are proved to be γγ-minimum.

1. INTRODUCTION

The concept of the inverse domination number was introduced by Kulli and Sigarkanti [3]. Let D be a dominating set (γ-set) of G. A dominating set D’ contained in V\D is called an inverse dominating set of G with respect to D. The smallest cardinality among all minimal dominating sets in V\D is called inverse domination number γ’(G).

In 2010, Tamizh Chelvam and Grace Perma [4] have characterized graphs with γ(G) = γ’(G) = (n-1)/2. Inverse domination number is already established for the following graphs: complete graphs, complete bi-partite graphs, cycles paths wheel graph, grid graph, corona graphs. In this article we have determined the inverse domination number for some graphs.

Hedetniemi et al. [2] defined and studied the disjoint domination number γγ(G) of a graph G. The existence of disjoint dominating sets has been studied by many researchers. In [1] V. Anusuya, R. Kala have already found disjoint dominating sets for the following graphs: wheel graph, helm graph, web graph, grid graph, star graph, trestled graph, paths, cycles. In this article we have found disjoint dominating sets for some graphs and proved them to be γγ-minimum.
2. INVERSE DOMINATION FOR SOME FAMILIES OF GRAPHS

2.1 Definition [6]
For any integer \( k \geq 1 \), let \( \mathbb{Z}_{3k-1} \) denote the additive group of integer modulo \( 3k-1 \) and let \( C \) be the subset of \( \mathbb{Z}_{3k-1} \) consisting of the elements congruent to 1 modulo 3. The Andrasfai graph is the cayley graph \( X(G,C) \) and it is denoted by \( \text{And}(k) \).
Let \( A_1 = \{ x \in V(G) / x \equiv 0 (\text{mod } 3) \} \), \( A_2 = \{ x \in V(G) / x \equiv 1 (\text{mod } 3) \} \), and \( A_3 = \{ x \in V(G) / x \equiv 2 (\text{mod } 3) \} \). Then \( |A_1| = k = |A_3| \) and \( |A_3| = k - 1 \).
Note that \( A_1, A_2, \) and \( A_3 \) are independent sets.

2.2 Theorem
\( \gamma'(\text{And}(k)) = 3 \) for \( k \geq 3 \).

Proof:
\( \text{And}(k) \) is a \( k \)-regular graph. Since it has no full degree vertex, \( \gamma > 1 \).
For \( u, v \in V(\text{And}(k)) \), \( d(u) + d(v) = 2k < 3k - 1 \), including \( u \) and \( v \), we have \( 2k + 2 < 3k - 1 \). Hence no two element set can be a dominating set. Therefore \( \gamma(\text{And}(k)) > 2 \).
Let \( D = \{0, 1, 2\} \), \( 0 \in A_1, 1 \in A_2, 2 \in A_3 \).
By construction of \( \text{And}(k) \), every member of \( A_2 \) is adjacent to 0, every member of \( A_3 \) is adjacent to 1 and every member of \( A_1 \) except 0 is adjacent to 2.
Therefore \( D \) is a minimum dominating set. \( D' = \{3, 4, 5\} \subseteq V \setminus D \). Therefore \( D' \) is an inverse dominating set with respect to \( D \).
Hence \( \gamma'(\text{And}(k)) = 3 \).

2.3 Definition
Prime complete graph \( PC_n \) on \( n \) vertices is a graph with \( V(PC_n) = \{1, 2, 3, ..., n\} \) and \( E(PC_n) = \{(i, j) / 1 \leq i \neq j \leq n, i, j \text{ are relatively prime}\} \).

2.4 Theorem \( \gamma'(PC_n) = 1 \).

Proof: Let \( PC_n \) be the prime complete graph with \( n \) vertices. Let \( V = \{1, 2, ..., n\} \) be the vertex set.
Let \( D = \{1\} \). Clearly it is a dominating set of \( PC_n \). Therefore, \( \gamma(PC_n) = 1 \).
\( V \setminus D = \{2, 3, ..., n\} \). By Bertrand’s postulate for every \( n > 1 \) there is always at least one prime \( p \) such that \( n < p < 2n \). So there exists a prime \( p \) between \( n/2 \) and \( n \) such that \( (p, i) = 1 \) for \( 1 \leq i \leq n \).
\{p\} \subseteq V \setminus D \) is an inverse dominating set with respect to \( D \). Hence \( \gamma'(PC_n) = 1 \).

2.5 Definition
Shadow graph \( Sh(G) \) is obtained by the adjunction of a new vertex \( v' \) for each vertex \( v \) in \( G \) and then joining \( v' \) to every neighbour of \( v \) in \( G \).

2.6 Theorem \( \gamma'(\text{Sh}(k_1,n)) = n + 1 \)

Proof. Let \( V(\text{Sh}(k_1,n)) = \{u_{i+1}/i = 0, 1, 2, ..., n\} \cup \{v_{i+1}/i = 0, 1, 2, ..., n\} \)
and \( E(\text{Sh}(k_1,n)) = \{u_1u_{i+1}, u_1v_{i+1}/i = 1, 2, ..., n\} \cup \{v_1u_{i+1}/i = 1, 2, ..., n\} \)
\( |V(\text{Sh}(k_1,n))| = 2n + 2 \). There is no full degree vertex. Therefore \( \gamma(\text{Sh}(k_1,n)) > 1 \).
Let \( D_1 = \{u_1, u_2\} \), \( u_1 \) is adjacent to \( u_{i+1} \) and \( v_{i+1} \) for \( i = 1, 2, ..., n \) and \( u_2 \) is adjacent to \( u_1 \) and \( v_1 \). Therefore \( D_1 = \{u_1, u_2\} \) is a minimum dominating set.
Hence $\gamma(Sh(k_1,n)) = 2$.

\[ V \setminus D_1 = \{v_1, v_2, ..., v_{n+1}\} \cup \{u_3, u_4, ..., u_n\}. \]

Let $S \subset V \setminus D_1$ be a dominating set of $Sh(k_1,n)$.

$v_2, v_3, ..., v_{n+1}$ are pendant vertices adjacent to $u_1$ alone and $u_1 \notin V \setminus D_1$.

Hence any dominating set contained in $V \setminus D_1$ should contain $v_2, v_3, ..., v_{n+1}$.

Note also that $u_3, u_4, ..., u_{n+1}$ are not adjacent to $v_i, i = 2, 3, ..., n$. So we need at least one member other than $v_2, v_3, ..., v_{n+1}$ to dominate these vertices. Hence $|S| \geq n + 1$.

Inverse dominating set with respect to $D_1$ should have at least $n + 1$ elements. Let $D_1' = \{v_i+1/ i=0,1,2,...,n\}$. $v_1$ is adjacent to $u_2, u_3, ..., u_{n+1}$. The pendant vertices $v_2, ..., v_{n+1}$ are adjacent to $u_1$. Therefore $D_1'$ is an inverse dominating set with respect to $D_1$. $Sh(k_1,n)$ has exactly two $\gamma$-sets namely $D_1 = \{u_1, u_2\}$ and $D_2 = \{u_1, v_1\}$.

$D_1' = \{v_1, v_2, ..., v_{n+1}\} \subseteq V \setminus D_1$ and it is a dominating set. It is also a minimum dominating set with respect to $D_1$. Clearly $V \setminus D_2$ does not contain any dominating set of cardinality less than $|D_1'|$. Hence $\gamma'(Sh(k_1,n)) = n + 1$.

2.7 Definition [6]
The windmill graph $W_d(k,n)$ is an undirected graph constructed for $k \geq 2$ and $n \geq 2$ by joining $n$ copies of the complete graph $K_k$ at a shared vertex.

2.8 Theorem $\gamma'(W_d(k,n)) = n$

Proof. Consider Windmill graph $W_d(k,n)$. It has $n$ copies of complete graph $K_k$ joined at a shared vertex. Let $u_0$ denote the shared vertex.

Number of vertices in $W_d(k,n) = (k-1)n + 1$.

Number of edges in $W_d(k,n) = \frac{nk(k-1)}{2}$.

Let $D = \{u_0\}$. Clearly it is a dominating set. There is no other full degree vertex in $W_d(k,n)$. Therefore $D$ is the unique minimum dominating set. Vertices of one copy except $u_0$ are not adjacent to vertices in any other copy. So we need at least one vertex in each copy in any inverse dominating set with respect to $D$.

Since each copy is a complete graph, it is enough to take one vertex from each copy.

Hence $\gamma'(W_d(k,n)) = n$.

2.9 Definition [5]
The $m$-book graph is defined as the graph cartesian product $S_{m+1} \times P_2$ where $S_{m+1}$ is a star graph and $P_2$ is the path graph on two nodes. The generalization of the book graph to $n$ stacked pages is the $(m,n)$-stacked book graph. It is denoted by $B_{n}^{k}$, where $n$ is the number of pages and each page is a $k$-sided polygon.

2.10 Theorem $\gamma'(B_{n}^{k}) = 1$.

Proof. Let $V(B_{n}^{3}) = \{u_0, u_1, ..., u_n\}$ and $E(B_{n}^{3}) = \{u_0v_1\} \cup \{u_0u_2, u_1u_2, ..., u_0u_n, u_1u_n\}$. Let $D = \{u_0\}$. Clearly it is a dominating set. Therefore $\gamma(B_{n}^{3}) = 1$.

$V \setminus D = \{u_1, u_2, ..., u_n\}$. $D = \{u_1\}$ is an inverse dominating set with respect to $D$.

Hence $\gamma'(B_{n}^{3}) = 1$. 

4159 ISSN: 2347-1697
International Journal of Informative & Futuristic Research (IJIFR)
Volume - 3, Issue - 11, July 2016
Continuous 35th Edition, Page No.: 4157-4166

S. M. Meena Rani, B. Eswari :: Pseudo Topogenic And Totally Null Topogenic Graphs
2.11 Theorem $\gamma'(B_n^4) = n$.

Proof. Let $V(B_n^4) = \{u_0, u_1, ..., u_n, v_0, v_1, ..., v_n\}$ and $E(B_n^4) = \{u_iv_i : i = 0, 1, ..., n\} \cup \{u_0u_i, v_0v_i : i = 1, 2, ..., n\}$.

Number of vertices in $B_n^4 = 2n + 2$.

Number of edges in $B_n^4 = 3n + 1$.

Let $D = \{u_0, v_0\}$. Clearly it is a dominating set. There is no vertex in $B_n^4$ with degree $2n + 1$. Therefore, $\gamma(B_n^4) > 1$. Hence $D$ is a minimum dominating set. $V - D = \{u_1, v_1, u_2, v_2, ..., u_n, v_n\}$. The set $D' = \{u_1, v_2, u_3, u_4, ..., u_{n-1}, u_n\}$. $u_1$ is dominated by $u_0$ and $v_1, v_2$ are dominated by $v_0$ and $u_2$. Then $u_i$ is dominated by $v_i$ for $i = 3, 4, ..., n$.

Therefore $D'$ is an inverse dominating set. In order to cover $u_i$ and $v_i$, one of them must be taken in any dominating set for $1 \leq i \leq n$. Note also that $\{u_0, v_0\}$ is the unique minimum dominating set. Hence $\gamma'(B_n^4) = n$.

2.12 Definition

Let $G_{2,n}$ denote the graph consisting of two cycles each of size $n$ with a shared edge.

Dominating sets as well as inverse dominating sets of $G_{2,n}$ are listed below.

For $n = 3$

$\gamma$-set $D = \{x\}$. Inverse dominating set $D' = \{y\}$

For $n = 4$

$\gamma$-set $D = \{x, y\}$. Inverse dominating set $D' = \{v_1, u_2\}$

For $n = 5$

$\gamma$-set $D = \{x, u_2, v_3\}$. Inverse dominating set $D' = \{u_1, y, v_2\}$

For $n = 6$
Figure 2.4: Inverse dominating set of two cycles with common edge for n=6
\( \gamma \)-set \( D = \{x, u_3, v_3\} \). Inverse dominating set \( D' = \{u_2, y, v_2\} \) for \( n = 7 \)

Figure 2.5: Inverse dominating set of two cycles with common edge for n=7
\( \gamma \)-set \( D = \{x, u_3, y, v_3\} \). Inverse dominating set \( D' = \{u_1, u_4, v_2, v_5\} \) for \( n = 8 \)

Figure 2.6: Inverse dominating set of two cycles with common edge for n=8
\( \gamma \)-set \( D = \{x, u_3, u_6, v_2, v_5\} \). Inverse dominating set \( D' = \{u_1, u_4, y, v_4, v_1\} \) for \( n = 9 \)

Figure 2.7: Inverse dominating set of two cycles with common edge for n=9
\( \gamma \)-set \( D = \{u_1, u_4, u_7, v_1, v_3, v_6\} \). Inverse dominating set \( D' = \{x, u_2, u_5, y, v_2, v_5\} \) for \( n = 10 \)
Figure 2.8: Inverse dominating set of two cycles with common edge for n=10
\( \gamma \)-set \( D = \{ x, u_3, u_6, v_3, v_6 \} \). Inverse dominating set \( D' = \{ u_1, u_4, u_7, v_2, v_5, v_8 \} \)
For \( n = 11 \)

Figure 2.9: Inverse dominating set of two cycles with common edge for n=11
\( \gamma \)-set \( D = \{ x, u_3, u_6, u_9, v_2, v_5, v_8 \} \).
Inverse dominating set \( D' = \{ u_1, u_4, u_7, y, v_1, v_4, v_7 \} \)
For \( n = 12 \)

Figure 2.10: Inverse dominating set of two cycles with common edge for n=12
\( \gamma \)-set \( D = \{ x, u_3, u_6, u_9, v_3, v_6, v_9 \} \).
Inverse dominating set \( D' = \{ u_2, u_5, u_8, y, v_2, v_5, v_8 \} \).
Hence we conclude that \( \gamma(G_{2,n}) = \gamma'(G_{2,n}) = \left\lfloor \frac{2n-2}{3} \right\rfloor \) if \( n \neq 0 \pmod{3} \) and
\[ \left\lfloor \frac{2n-2}{3} \right\rfloor \] if \( n \equiv 0 \pmod{3} \).

2.13 Theorem \( \gamma'(\text{Tetrahedron}) = 1 \)
Proof. Tetrahedron is a 3-regular graph with 4 vertices. Let \( D = \{ 1 \} \).
Clearly it is a dominating set. Therefore \( \gamma(\text{Tetrahedron}) = 1 \).
\( V \setminus D = \{ 2, 3, 4 \} \). \( D' = \{ 2 \} \) is an inverse dominating set with respect to \( D \). Hence
\( \gamma'(\text{Tetrahedron}) = 1 \).
2.14 Theorem $\gamma'(\text{Cube}) = 2$

**Proof.** Cube is a 3-regular graph with 8 vertices. Let $D = \{1, 6\}$. Clearly it is a dominating set. For any Graph $G$, \[
\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n-\Delta(G) \rceil \leq 2.
\]

Hence $\gamma(\text{Cube}) \geq 2$. Hence $\gamma(\text{Cube}) = 2$.

$V \setminus D = \{2, 3, 4, 5, 7, 8\}$. $D' = \{2, 7\}$ is an inverse dominating set with respect to $D$. Hence $\gamma'(\text{Cube}) = 2$.

![Figure 2.11: Inverse dominating set of Cube](image)

2.15 Theorem $\gamma'(\text{Octahedron}) = 2$

**Proof.** Octahedron is a 4-regular graph with 6 vertices. Let $D = \{1, 6\}$. Clearly it is a dominating set. There is no vertex in octahedron with degree 5. Therefore $\gamma(\text{Octahedron}) > 1$.

$D$ is a minimum dominating set. $V \setminus D = \{2, 3, 4, 5\}$. Any two vertices of $V \setminus D$ is an inverse dominating set. Hence $\gamma'(\text{Octahedron}) = 2$.

![Figure 2.12: Inverse dominating set of Octahedron](image)

2.16 Theorem $\gamma'(\text{Icosahedron}) = 2$

**Proof.** Icosahedron is a 5-regular graph with 12 vertices. Let $D = \{1, 4\}$. Clearly it is a dominating set. For any Graph $G$, \[
\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n-\Delta(G) \rceil \leq 2.
\]

$\gamma(\text{Icosahedron}) \geq 2$.

Therefore $D$ is a minimum dominating set. Hence $\gamma(\text{Icosahedron}) = 2$.

$V \setminus D = \{2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$. $D' = \{7, 10\}$ is an inverse dominating set with respect to $D$. Hence $\gamma'(\text{Icosahedron}) = 2$.

![Figure 2.13: Inverse dominating set of Icosahedron](image)
3. DISJOINT DOMINATING SETS

3.1 Definition
We say that a graph $G$ is $\gamma\gamma$-minimum if it has two disjoint dominating sets ($\gamma$-sets), that is, $\gamma\gamma(G) = 2\gamma(G)$. Similarly, a graph $G$ is called $\gamma\gamma$-maximum if $\gamma\gamma(G) = n$.

3.2 Theorem
Prime complete graphs $PC_n$ are $\gamma\gamma$-minimum.

Proof. By theorem 3.4, we have $\gamma(PC_n) = 1 = \gamma'(PC_n)$ and $\gamma\gamma(PC_n) = 2 = 2\gamma(PC_n)$. Hence $PC_n$ are $\gamma\gamma$-minimum.

3.3 Theorem
Andrasfai graphs $And(k)$ are $\gamma\gamma$-minimum

Proof. By theorem 3.2, we have $\gamma(And(k)) = 3 = \gamma'(And(k))$ Therefore $\gamma\gamma(And(k)) = 6 = 2\gamma(And(k))$, Hence $And(k)$ are $\gamma\gamma$-minimum.

3.4 Theorem
Book graphs $B_n^3$ are $\gamma\gamma$-minimum.

Proof. By theorem 3.7, we have $\gamma(B_n^3) = 1 = \gamma'(B_n^3)$. Therefore $\gamma\gamma(B_n^3) = 2 = 2\gamma(B_n^3)$. Therefore $B_n^3$ are $\gamma\gamma$-minimum.

3.5 Theorem
Books with pentagonal pages $B_n^5$ are $\gamma\gamma$-minimum.

Proof. Let $V(B_n^5) = \{v_1, v_2, ..., v_{3n+2}\}$ and $E(B_n^5) = \{v_1v_3r+2/r = 1, 2, ..., n\} \cup \{v_2v_3r/r = 1, 2, ..., n\} \cup \{v_3r v_3r+1, v_3r+1v_3r+2/r = 1, 2, ..., n\}$.

It can be easily verified that $\gamma(B_n^5) = n+1$. Let $D = \{1, 4, 7, ..., 3n + 1\}$ and $D' = \{2, 5, 8, ..., 3n + 2\}$. $D'$ is also a $\gamma$-set.

$D$ and $D'$ are disjoint minimum dominating sets. Therefore $\gamma\gamma(B_n^5) = 2n+2 = 2\gamma(B_n^5)$.

Hence $B_n^5$ are $\gamma\gamma$-minimum.
3.6 Theorem  Let $G_n$ denote the graph consisting of two cycles each of size $n$ with a shared vertex. Then

1. $\gamma(G_n) = 1 + \lceil \frac{n-3}{3} \rceil$

2. $\gamma'(G_n) = 1 + \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-1}{3} \rceil$

3. $G_n$ is $\gamma\gamma$-minimum iff $n \equiv 1 \pmod{3}$

Proof.  Let $V(G_n) = \{x, u_i, v_i / i=1,2,\ldots,n-1\}$ and $E(G_n)=\{u_iu_{i+1}, v_iv_{i+1} / i=1,2,\ldots,n-2\} \cup \{xu_1, u_{n-1}, x, v_1, v_{n-1}\}$. Let $D$ be a dominating set.

Case 1: Let $x \in D$.

$x$ covers four vertices namely $u_1$, $u_{n-1}$, $v_1$, $v_{n-1}$. $G_n' = G_n \setminus \{x, u_1, u_{n-1}, v_1, v_{n-1}\}$ is the disjoint union of two paths each with $n-3$ vertices. $\gamma(G_n') = 2 \lceil \frac{n-3}{3} \rceil$.

Therefore $|D| \geq 1 + 2 \lceil \frac{n-3}{3} \rceil$. (1)

Case 2: Let $x \not\in D$.

$X$ should be covered by one member from $\{u_1,u_{n-1},v_1,v_{n-1}\}$. Without loss of generality, let $u_1 \in D$. Consider $G_n' = G_n \setminus \{x, u_1, u_2\}$. $G_n'$ is the disjoint union of two paths $u_3 \ldots u_{n-1}$ and $v_1 \ldots v_{n-1}$.

$\gamma(G_n') = \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-1}{3} \rceil$. Therefore $|D| \geq 1 + \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-1}{3} \rceil$. (2)

From (1) and (2), it follows that any minimum dominating set should contain $x$ and $\gamma(G_n) = 1 + 2 \lceil \frac{n-3}{3} \rceil$. Inverse dominating set does not contain $x$ and hence $\gamma(G_n') = 1 + \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-1}{3} \rceil$.

Claim: $\lceil \frac{n-3}{3} \rceil = \lceil \frac{n-1}{3} \rceil \iff n \equiv 1 \pmod{3}$

Proof:

Subclaim 1: $n \equiv 0 \pmod{3}$

Let $n = 3m$.

$\lceil \frac{n-3}{3} \rceil = \lceil \frac{3m-3}{3} \rceil = m-1$

$\lceil \frac{n-1}{3} \rceil = \lceil \frac{3m-1}{3} \rceil = m$

Subclaim 2: $n \equiv 1 \pmod{3}$

Let $n = 3m+1$.

$\lceil \frac{n-3}{3} \rceil = \lceil \frac{3m+1-3}{3} \rceil = m$
\left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{3m+1-1}{3} \right\rfloor = m

**Subclaim 3:** \( n \equiv 2 \pmod{3} \)
Let \( n = 3m+2 \).
\left\lfloor \frac{n-3}{3} \right\rfloor = \left\lfloor \frac{3m+2-3}{3} \right\rfloor = m
\left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{3m+2-1}{3} \right\rfloor = m

Hence \( \left\lfloor \frac{n-3}{3} \right\rfloor = \left\lfloor \frac{n-1}{3} \right\rfloor \iff n \equiv 1 \pmod{3} \).
Therefore \( \gamma(G^n) = \gamma(G^n') \iff n \equiv 1 \pmod{3} \).
Hence \( G^n \) is \( \gamma \gamma \)-minimum iff \( n \equiv 1 \pmod{3} \).

### 6. BIOGRAPHIES

**Dr. S.M.Meena Rani** is working as an Associate Professor in Mathematics in V.V.Vanniaperumal College for Women, Virudhungar. She is having 30 years of teaching experience and 7 years of research experience. She has published more than 7 papers in international research journals. Her field of research is Graph Theory and Neural Network.

**R.Akila** is a M.Phil Scholar in the department of Mathematics, V.V.Vanniaperumal College for Women, Virudhungar. Her field of research is Graph Theory.